

## *Factorization of Symmetric Expressions*

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*Let us factorize the following symmetric expression  $a^3 + b^3 + c^3 - 3abc$ .  
We usually teach it in our mathematics class as follows:*

$$\begin{aligned}
 a^3 + b^3 + c^3 - 3abc &= (a^3 + b^3) + c^3 - 3abc \\
 &= \{(a + b)^3 - 3ab(a + b)\} + c^3 - 3abc \\
 &= \{(a + b)^3 + c^3\} - \{3ab(a + b) + 3abc\} \\
 &= (a + b + c)\{(a + b)^2 - (a + b)c + c^2\} - 3ab(a + b + c) \\
 &= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)
 \end{aligned}$$

*This method is easy but rather elaborate.*

*We know that the symmetric expressions are able to represent in the form of polynomials by using its elementary symmetric expressions. Using this theorem, we can factorize it as follows;*

*Suppose  $f(a, b, c) = a^3 + b^3 + c^3 - 3abc$ , and we substitute  $c = 0$ , we find*

$$\begin{aligned}
 f(a, b, 0) &= a^3 + b^3 \\
 &= (a + b)^3 - 3ab(a + b) \dots\dots\dots (1)
 \end{aligned}$$

*We substitute ternary elementary symmetric expressions  $a + b + c$ ,  $ab + bc + ca$ ,  $abc$  for binary ones  $a + b$ ,  $ab$  respectively and we suppose  $r(a, b, c)$  is a difference:*

$$\begin{aligned}
 r(a, b, c) &= f(a, b, c) - (a + b + c)^3 + 3(ab + bc + ca)(a + b + c) + kabc \\
 &= kabc, \quad \text{where } k \text{ is constant.} \\
 r(a, b, c) &= 0 \quad \text{where } k = 0.
 \end{aligned}$$

*By using (2) we find*

$$\begin{aligned}
 0 &= f(a, b, c) - (a + b + c)^3 + 3(a + b + c)(ab + bc + ca) \\
 f(a, b, c) &= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)
 \end{aligned}$$

*This method is rather difficult for high school students, but has a lot of application.*

*However, we can factorize this symmetric expression as follows:*

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a + by + cy^2)(a + by^2 + cy)$$

*where  $y$  is a cubic root of 1.*

*The quadratic equation, where  $a$  is to be found,*

$$a^2 + b^2 + c^2 - ab - bc - ca = 0$$

*that is to say,*

$$a^2 - (b + c)a + (b^2 - bc + c^2) = 0$$

*This has following solutions:*

$$\begin{aligned} a &= c - (b - c)y, \quad c - (b - c)y^2 \\ &= -by - cy^2, \quad -by^2 - cy \end{aligned}$$

*Therefore,*

$$\begin{aligned} a^2 + b^2 + c^2 - ab - bc - ca &= (a + b + c)(a + by + cy^2)(a + by^2 + cy) \\ f(a, b, c) &= (a + b + c)(a + by + cy^2)(a + by^2 + cy) \end{aligned}$$

*This method is often used by high school students. The next one is rather difficult but simpler.*

$$f(a, b, c) = f(a, by, cy^2) = f(a, by^2, cy)$$

*As  $f(a, b, c)$  has a factor  $a + b + c$ , it also has factors  $a + by + cy^2$  and  $a + by^2 + cy$ .*

$$f(a, b, c) = (a + b + c)(a + by + cy^2)(a + by^2 + cy)$$

*The cubic equation  $f(x, b, c) = 0$ ; that is,  $x^3 - 3bcx + (b^3 + c^3) = 0$  is able to solve algebraically and its solutions are as follows:*

$$x = -b - c, \quad -by - cy^2, \quad -by^2 - cy \quad (\text{Cardano's formula})$$

*2) Let us factorize the following symmetric expression*

$$a^4 + b^4 + c^4 + d^4 - 2(a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^2) - 8abcd$$

*Suppose*

$$f(a, b, c, d) = a^4 + b^4 + c^4 + d^4 - 2(a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^2) - 8abcd.$$

*and we substitute  $d = 0$ , we find*

$$\begin{aligned} f(a, b, c, 0) &= a^4 + b^4 + c^4 - 2(a^2b^2 + a^2c^2 + b^2c^2) \\ &= (a + b + c)^4 - 4(a + b + c)^2(ab + bc + ca) + 8abc(a + b + c) \end{aligned}$$

*We substitute ternary elementary symmetric expressions*

$a + b + c + d$ ,  $ab + ac + ad + bc + bd + cd$ ,  $abc + abd + acd + bcd$ ,  $abcd$   
for binary ones  $a + b + c$ ,  $ab + bc + ca$ ,  $abc$  respectively and we suppose

$r(a, b, c, d)$  is a difference,

$$\begin{aligned} r(a, b, c, d) &= f(a, b, c, d) - (a + b + c + d)^4 \\ &\quad + 4(a + b + c + d)^2(ab + ac + ad + bc + bc + bd + cd) \\ &\quad - 8(abc + abd + acd + bcd)(a + b + c + d) + kabcd \\ &= (k - 16)abcd. \dots\dots\dots(3) \end{aligned}$$

$$r(a, b, c, d) = 0 \quad \text{when } k = 16$$

By using (3) we find

$$\begin{aligned} 0 &= f(a, b, c, d) - (a + b + c + d)^4 \\ &\quad + 4(a + b + c + d)^2(ab + ac + ad + bc + bd + cd) \\ &\quad - 8(abc + abd + acd + bcd)(a + b + c + d) + 16abcd \\ f(a, b, c, d) &= (a + b + c + d)^4 \\ &\quad - 4(a + b + c + d)^2(ab + ac + ad + bc + bd + cd) \\ &\quad + 8(abc + abd + acd + bcd)(a + b + c + d) - 16abcd \end{aligned}$$

Suppose  $a + b + c + d = t$ , we find

$$f(a, b, c, d) = -( - a + b + c + d)(a - b + c + d)(a + b - c + d)(a + b + c - d)$$

$f(x, b, c, d) = 0$ , that is

$$x^4 - 2(b^2 + c^2 + d^2)x^2 - 8bcdx + (b^4 + c^4 + d^4) - 2(b^2c^2 + b^2d^2 + c^2d^2) = 0$$

*Remark: We can factorize this quartic expression by Euler's method.*

### Reference Books

数と式 (代数入門)	1990	遠山 啓 著	講談社
初等代数学	昭和41年	成田 正雄 著	共立出版
代数学講義	昭和5年	高木 貞治 著	共立出版

*Supplement*

$$\begin{aligned}
r(a, b, c, d) &= a^4 + b^4 + c^4 + d^4 - 2(a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^2) \\
&\quad - 8abcd - (a + b + c + d)^4 \\
&\quad + 4(a + b + c + d)^2(ab + ac + ad + bc + bd + cd) \\
&\quad - 8(abc + abd + acd + bcd)(a + b + c + d) + kabcd
\end{aligned}$$

$$\begin{aligned}
a^4 + b^4 + c^4 + d^4 &= (a^2 + b^2 + c^2 + d^2)^2 - 2(a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^2) \\
&= (a + b + c + d)^4 - 4(a + b + c + d)^2(ab + ac + ad + bc + bd + cd) \\
&\quad + 2(ab + ac + ad + bc + bd + cd)^2 \\
&\quad + 4(a + b + c + d)(abc + abd + acd + bcd) - 4abcd
\end{aligned}$$

$$\begin{aligned}
a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^2 \\
&= (ab + ac + ad + bc + bd + cd)^2 \\
&\quad - 2(a + b + c + d)(abc + abd + acd + bcd) + 2abcd
\end{aligned}$$

$$\begin{aligned}
r(a, b, c, d) &= (a + b + c + d)^4 - 4(a + b + c + d)^2(ab + ac + ad + bc + bd + cd) \\
&\quad + 4(ab + ac + ad + bc + bd + cd)^2 - 2(ab + ac + ad + bc + bd + cd)^2 \\
&\quad + 4(a + b + c + d)(ab + ac + ad + bc + bd + cd) - 4abcd \\
&\quad - 2(ab + ac + ad + bc + bd + cd)^2 + 4(a + b + c + d)(ab + ac + ad + bc + bd + cd) \\
&\quad - 4abcd - 8abcd - (a + b + c + d)^4 \\
&\quad + 4(a + b + c + d)^2(ab + ac + ad + bc + bd + cd) \\
&\quad - 8(abc + abd + acd + bcd)(a + b + c + d) + kabcd \\
&= (k - 16)abcd
\end{aligned}$$