



diagonal equal to  $LG$  and with shorter side  $LN$ .

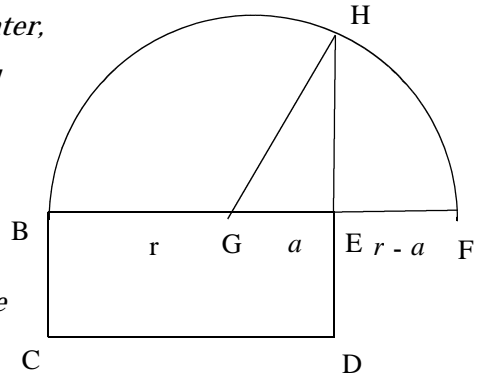
Then the longer side  $GN$  of this rectangle is the side of square desired.

Remark: This construction resembles to the Elements of Euclid Book II Proposition 14 ; To construct a square equal to a given rectilinear figure.

Let  $A$  be the given rectilinear figure  $A$ ; it is required to construct a square equal to the rectilinear figure  $A$ . For let there be constructed the rectangular parallelogram  $BD$  equal to the rectilinear figure  $A$ .

Then if  $BE$  is equal to  $ED$ , the square  $BD$  is desired one. But if not, one of the straight lines  $BE, ED$  is greater. Let  $BE$  be greater, and let it be produced to  $F$ ; let  $EF$  be made equal to  $ED$ , and let  $BF$  be bisected at  $G$ .

Describe semicircle with center  $G$  and distance  $BG$ , let  $DE$  be produced to  $H$ , and let  $GH$  be joined. Rectangular  $BD$  together with the square on  $GE$  is equal to the square on  $GH$ .



$$(r + a)(r - a) + a^2 = r^2$$

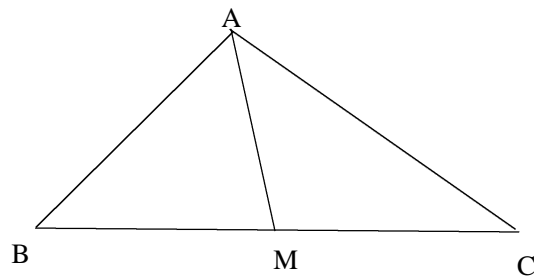
Rectangular  $BD$  together with the square on  $GE$  is equal to the squares on  $HE, GE$ . Let the square on  $GE$  be subtracted from each, therefore the rectangular is equal to the square on  $HE$ . Therefore a square on  $HE$  is equal to the given rectilinear figure  $A$ .

## 2 Extension of the theorem

### (1) Pappus' theorem

If  $M$  be the middle point of the side  $BC$  in a triangle  $ABC$

$$AB^2 + AC^2 = 2(BM^2 + AM^2)$$



This theorem is an extension of the Pythagorean theorem as follows:

When  $A = R$ ,  $AM = BM$

$$2(BM^2 + AM^2) = 4BM^2 = 4\left(\frac{BC}{2}\right)^2 = BC^2$$

$$AB^2 + AC^2 = BC^2$$

(2) *Thabit ibn - Qurra's theorem* <sup>(1)</sup>

If from vertex  $A$  of any triangle  $ABC$  one draws lines intersecting  $BC$  in points  $B'$ ,  $C'$  such that angles  $AB'B$ ,  $AC'C$  are each equal to angle  $A$ , then

$$AB^2 + AC^2 = BC(BB' + CC')$$

*Proof:*  $\triangle ABC \sim \triangle ABB'$ ,  $\triangle ABC \sim \triangle ACC'$

$$AB : BC = BB' : AB \quad AB^2 = BC \cdot BB'$$

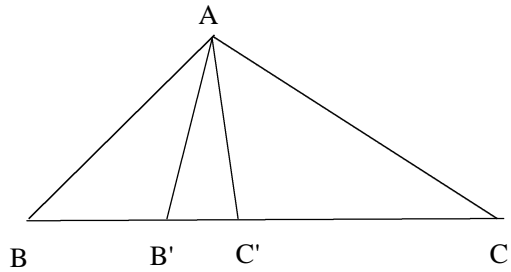
$$AC : CC = BC : AC \quad AC^2 = BC \cdot CC'$$

$$AB^2 + AC^2 = BC(BB' + CC')$$

When  $A = R$ ,  $B'$  coincides with  $C'$ .

Since  $BB' + CC' = BC$ ,

$$AB^2 + AC^2 = BC^2 \quad (\text{Pythagorean theorem})$$



*Remark:* When angle  $A$  is acute, this theorem holds.

(3) *Cosine theorem*

Here we designate the measures of three angles in  $\triangle ABC$  as  $A, B, C$  and the sides opposite these angles as  $a, b$ , and  $c$ , respectively.

$$a^2 = b^2 + c^2 - 2bc \cdot \cos A$$

When  $A = R$ ,  $a^2 = b^2 + c^2$

(4) For the inscribing quadrangle  $ABCD$  in a circle, we have

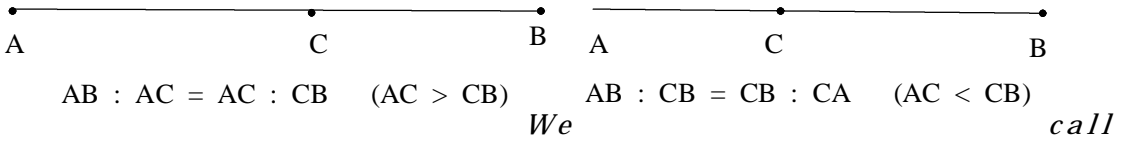
$$AB \cdot CD + BC \cdot AD = AC \cdot BD. \quad (\text{Ptolemy's theorem})$$

When  $A = B = R$ ,  $AB^2 + BC^2 = AC^2$

### 3 Golden section

When the line  $AB$  is given and divided at a point  $C$  in the following way:

whole line : larger segment = larger segment : shorter segment



that the line is divided in extreme and mean ratio (EMR). (The Elements of Book VI Definition 3)

And this division is called DEMR, another term is Golden section.

4 Golden section

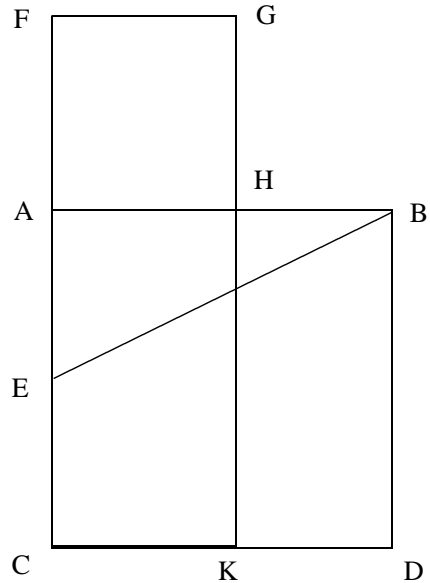
The Elements of Book II Proposition 11

To cut a given straight line so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining segment.

- 1) Let AB be the given straight line; let AC be bisected at the point E, and let BE be joined.
- 2) Let CA be drawn through to F, and let EF be made equal to BE.
- 3) Let the square FH be described on AF, and let GH be drawn through to K.

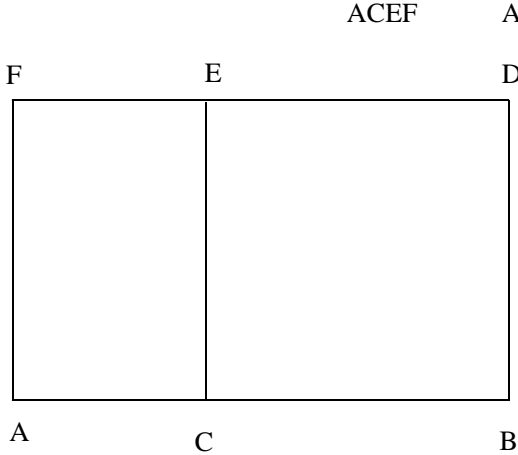
The line AB be cut at H so as to make the rectangle contained by AB, BH equal to the square on HA.

$$\begin{aligned}
 AHGF &= AH^2 \\
 HKDB &= DB \cdot HB = AB \cdot HB \\
 AHG &= HKDB \\
 AH^2 &= AB \cdot HB \\
 AH : HB &= AB : AH
 \end{aligned}$$



Gnomon's theorem <sup>7)</sup>

Suppose line AB which has been divided in DEMR at point C with BC being the larger segment, and then form the rectangle ABDF with DB = BC. Then,



Since AB is divided in DEMR at point C,

$$BC : CA = AB : BC$$

Since BC = DB = AF

$$AF : CA = AB : DB$$

ACEF      ABDF

### 5 Golden section in pentagon

Given regular polygon ABCDE, and draw the five diagonals, these diagonals intersect in points A',B',C',D',E', which form another pentagon. The diagonal points A',B',C',D',E', divide the diagonals in DEMR.

In the Fig,1

ABD'      ABC

$$AD' : AB = AB : AC$$

Since AD' = E'C, AB = AE',

$$E'C : AE' = AE' : AC$$

Point E' divides line AC in DEMR.

Similarly, the points A',B',C',D' divide the lines

BD,CE,DA,EB in DEMR.

Let AB = 1, and AC = x,

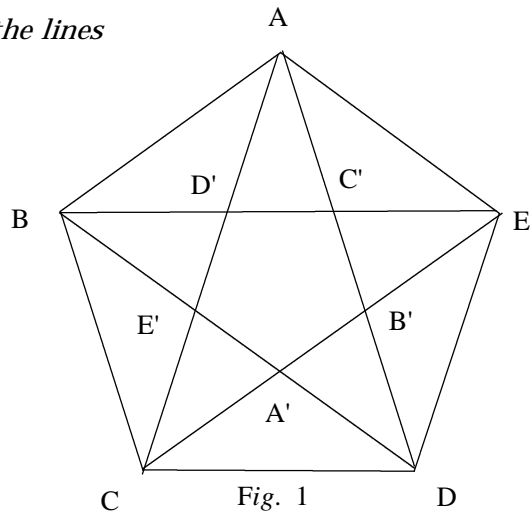
$$AC : AE' = AE' : E'C$$

$$AE' = x, E'C = AC - AE' = x - 1$$

$$x : 1 = 1 : x - 1$$

$$x^2 - x = 1 \quad (x > 0)$$

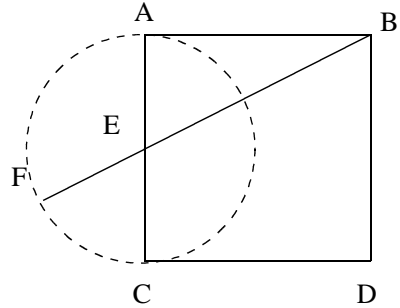
$$x = \frac{1 + \sqrt{1+4}}{2} = \frac{1 + \sqrt{5}}{2}$$



Using this fact, we can draw a pentagon as follows,

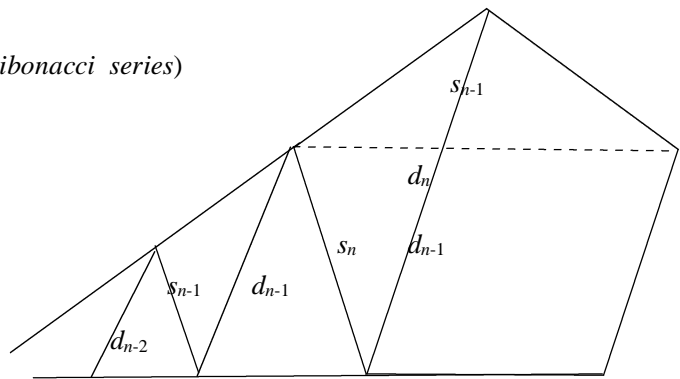
### 6 Construction of pentagon

- 1) Draw a square  $ABCD$  whose side is  $a$ .
- 2) Bisect  $AC$  at the point  $E$  and with the center and distance  $AE$ ; let the circle be described
- 3) Construct the regular pentagon with the side  $AB$ , and diagonal  $BF$ .



### 7 Heller's theorem

$$d_n = d_{n-1} + s_{n-1}, \quad d_{n-1} = s_n \quad (\text{Fibonacci series})$$



### References

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