

Pythagorean Theorem and Golden Section

December 31, 2004

1 The origin of the theorem

According to Carl B. Boyer, the Pythagorean theorem dates back to the Old Mesopotamia, "The Pythagorean theorem, for example, does not appear in any form in surviving documents from Egypt, but tablets even from the Old Babylonian period show that in Mesopotamia, the theorem was widely used." ¹⁾ This theorem also was written in the old Indian lore all in verse, Sulvasutra (the 8th century B.C. - the 2nd century A.D.). "India, like Egypt had its rope-strtchers, and the primitive geometrical lore acquired in connection with the laying out of temples and the measurement and constructing of altars, took the form of the body of knowledge known as the Sulvasutras, or rules of the cord." ¹⁾

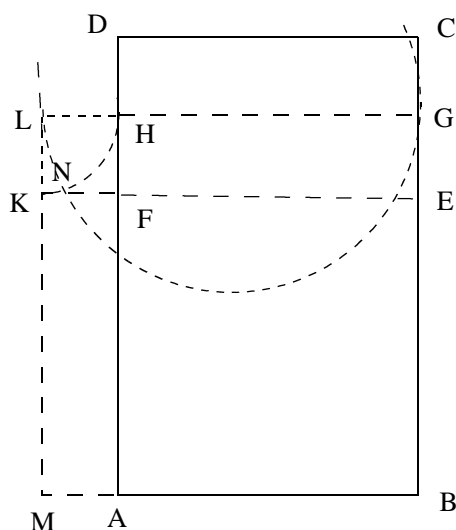
(1) *In this primitive account, we find rules for the construction of right angles by means of triples of cords such as 3, 4, and 5, 5, 12, and 13, 8, 15, and 17, or 12, 35, and 37.*

(2) The following construction shows us the Pythagorean theorem. To construct a square equal in area to the rectangle $ABCD$, lay off the shorter sides on the longer so that

$$AF = AB = BE = CD$$

and draw HG bisecting segments CE and DF ;
 extend EF to K , GH to L , and AB to M so that

$FK = HL = FH = AM$, and draw LKM . Now construct a rectangle with



diagonal equal to LG and with shorter side LN .

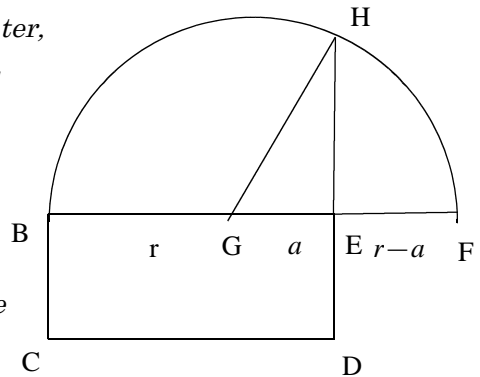
Then the longer side GN of this rectangle is the side of square desired.

Remark: This construction resembles to the *Elements of Euclid Book II Proposition 14* ; To construct a square equal to a given rectilineal figure.

Let A be the given rectilineal figure A ; it is required to construct a square equal to the rectilineal figure A . For let, there be constructed the rectangular parallelogram BD equal to the rectilineal figure A .

Then if BE is equal to ED , the square BD is desired one. But if not, one of the straight lines BE , ED is greater. Let BE be greater, and let it be produced to F ; let EF be made equal to ED , and let BF be bisected at G .

Describe semicircle with center G and distance BG , let DE be produced to H , and let GH be joined. Rectangular BD together with the square on GE is equal to the square on GH .



$$(r + a)(r - a) + a^2 = r^2$$

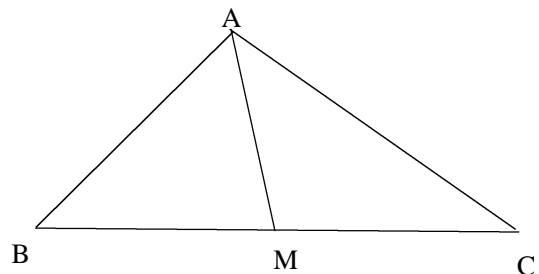
Rectangular BD together with the square on GE is equal to the squares on HE , GE . Let the square on GE be subtracted from each, therefore the rectangular is equal to the square on HE . Therefore a square on HE is equal to the given rectilineal figure A .

2 Extension of the theorem

(1) Pappus' theorem

If M be the middle point of the side BC in a triangle ABC

$$AB^2 + AC^2 = 2(BM^2 + AM^2)$$



This theorem is an extension of the Pythagorean theorem as follows:

When $\angle A = \angle R$, $AM = BM$

$$\therefore 2(BM^2 + AM^2) = 4BM^2 = 4\left(\frac{BC}{2}\right)^2 = BC^2$$

$$\therefore AB^2 + AC^2 = BC^2$$

(2) *Thabit ibn - Qurra's theorem* ⁽¹⁾

If from vertex A of any triangle ABC one draws lines intersecting BC in points B' , C' such that angles $AB'B$, $AC'C$ are each equal to angle A , then

$$AB^2 + AC^2 = BC(BB' + CC')$$

Proof: $\triangle ABC \sim \triangle ABB'$, $\triangle ABC \sim \triangle ACC'$

$$AB : BC = BB' : AB \quad \therefore AB^2 = BC \cdot BB'$$

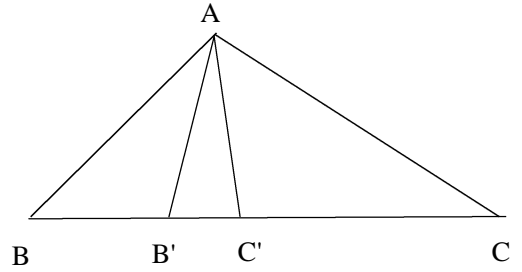
$$AC : CC' = BC : AC \quad \therefore AC^2 = BC \cdot CC'$$

$$\therefore AB^2 + AC^2 = BC(BB' + CC')$$

When $\angle A = \angle R$, B' coincides with C' .

Since $BB' + CC' = BC$,

$$AB^2 + AC^2 = BC^2 \quad (\text{Pythagorean theorem})$$



Remark: When angle A is acute, this theorem holds.

(3) *Cosine theorem*

Here we designate the measures of three angles in $\triangle ABC$ as A , B , C and the sides opposite these angles as a , b , and c , respectively.

$$a^2 = b^2 + c^2 - 2bc \cdot \cos A$$

$$\text{When } \angle A = \angle R, \quad a^2 = b^2 + c^2$$

(4) *For the inscribing quadrangle ABCD in a circle, we have*

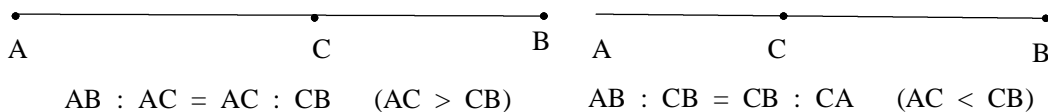
$$AB \cdot CD + BC \cdot AD = AC \cdot BD. \quad (\text{Ptolemy's theorem})$$

$$\text{When } \angle A = \angle B = \angle R, \quad AB^2 + BC^2 = AC^2$$

3 Golden section

When the line AB is given and divided at a point C in the following way:

$$\text{whole line} : \text{larger segment} = \text{larger segment} : \text{shorter segment}$$



We call that the line is divided into extreme and mean ratio (*EMR*). (*The Elements of Book VI Definition 3*)

And this division is called *DEMR*, another term is *Golden section*.

4 Golden section

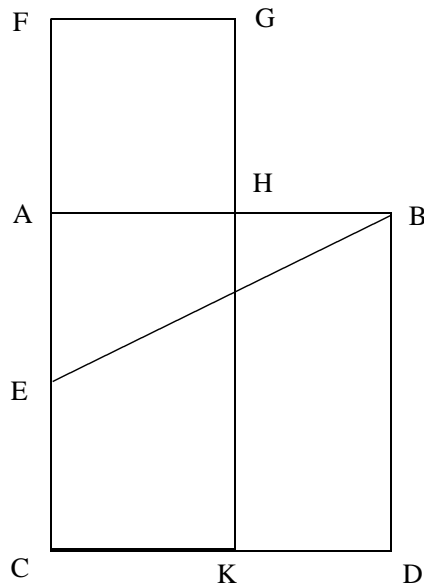
The Elements of Book II Proposition 11

To cut a given straight line so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining segment.

- 1) Let AB be the given straight line; let AC be bisected at the point E , and let BE be joined.
- 2) Let CA be drawn through to F , and let EF be made equal to BE .
- 3) Let the square FH be described on AF , and let GH be drawn through to K .

The line AB be cut at H to make the rectangle contained by AB , BH equal to the square on HA .

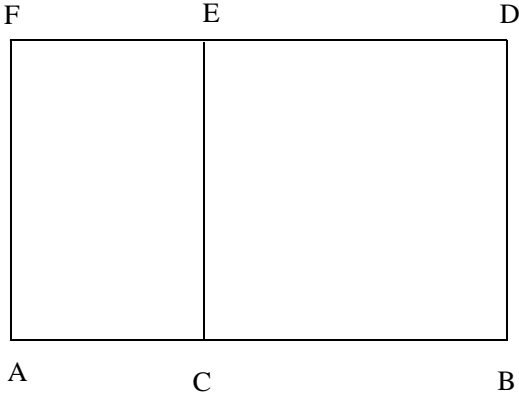
$$\begin{aligned}
 \square AHGF &= AH^2 \\
 \square HKDB &= DB \cdot HB = AB \cdot HB \\
 \square AHG &= \square HKDB \\
 AH^2 &= AB \cdot HB \\
 \therefore AH : HB &= AB : AH
 \end{aligned}$$



Gnomon's theorem ⁷⁾

Suppose line AB has been divided in DEMR at point C with BC being the larger segment, and then form the rectangle $ABDF$ with $DB = BC$. Then,

$$\square ACEF \sim \square ABDF$$



\therefore Since AB is divided in DEMR at point C ,

$$BC : CA = AB : BC$$

Since $BC = DB = AF$

$$AF : CA = AB : DB$$

$$\therefore \square ACEF \sim \square ABDF$$

5 Golden section in pentagon

Given regular polygon $ABCDE$, and draw the five diagonals, these diagonals intersect in points A', B', C', D', E' , which form another pentagon. The diagonal points A', B', C', D', E' , divide the diagonals in DEMR.

In the Fig,1 $\triangle ABD' \sim \triangle ABC$

$$\therefore AD' : AB = AB : AC$$

Since $AD' = E'C$, $AB = AE'$,

$$\therefore E'C : AE' = AE' : AC$$

\therefore Point E' divides line AC in DEMR.

Similarly, the points A', B', C', D' divide the lines BD, CE, DA, EB in DEMR.

Let $AB = 1$, and $AC = x$,

$$AC : AE' = AE' : E'C$$

$$AE' = x, E'C = AC - AE' = x - 1$$

$$\therefore x : 1 = 1 : x - 1$$

$$x^2 - x = 1 \quad (x > 0)$$

$$x = \frac{1 + \sqrt{1+4}}{2} = \frac{1 + \sqrt{5}}{2}$$

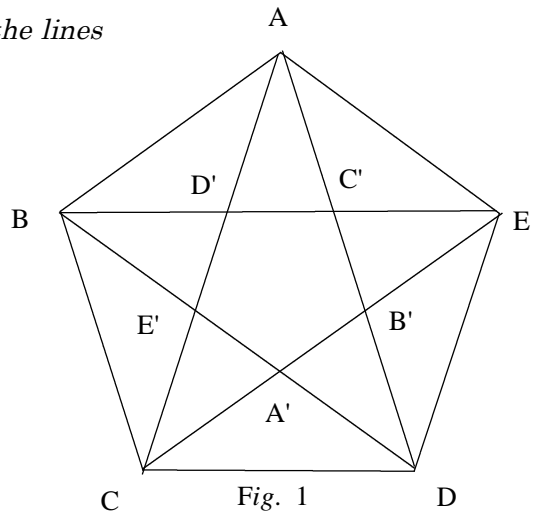
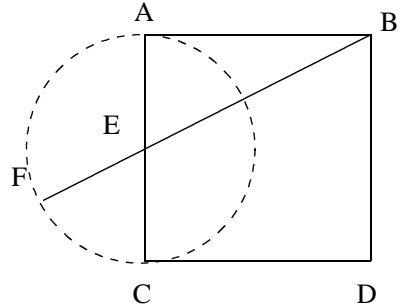


Fig. 1

Using this fact, we can draw a pentagon as follows,

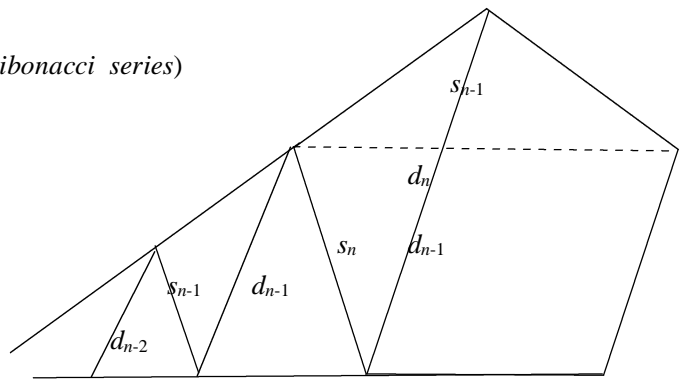
6 Construction of pentagon

- 1) Draw a square $ABCD$ whose side is a .
- 2) Bisect AC at the point E and with the center and distance AE ; let the circle be described
- 3) Construct the regular pentagon with the side AB , and diagonal BF .



7 Heller's theorem

$$d_n = d_{n-1} + s_{n-1}, \quad d_{n-1} = s_n \quad (\text{Fibonacci series})$$



References

- 1 Translates, Koshiro Nakamura, Eiko Terasaka, Shuntaro Ito, Mie Ikeda, *Euclid's the Elements*, Tokyo, Kyoritsu-Shuppan, 1971
2. Thomas L. Heath, *EUCLID THE ELEMENTS Vol.1, Vol.3*, New York, Dover, 1956
3. Kenichi Sato, *The Text Book of Mathematics History*, Toyo-Shoten, 1970
4. Carl B. Boyer, *A History of Mathematics*, New York, Wiley, 1968
5. Thomas L. Heath, *EUCLID THE ELEMENTS Vol.1, Vol.3*, New York, Dover, 1956
6. *Dictionary of Mathematics and Edition*, New York, McGraw-Hill, 2003
7. R. Herz-Fishler, *Mathematical History of the Golden Number*, N.Y., Dover, 1998