

# Euler's three Formulas

## Gamma, Beta and Zeta Functions (2007)

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### 1. Euler's Constant

*Theorem 1. The following series converges.*

$$1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \log n \quad (n > 0)$$

*Proof.* By putting  $c_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \log n$

$$c_{n+1} - c_n = \frac{1}{n+1} - \log \frac{n+1}{n}$$

$$= \frac{1}{n+1} - \int_n^{n+1} \frac{1}{x} dx$$

$$\int_n^{n+1} \frac{1}{x} dx = \text{PRSQ}$$

$$\frac{1}{n+1} = \text{TRSQ}$$

$$\frac{1}{n+1} - \int_n^{n+1} \frac{1}{x} dx < 0$$

$c_n$  is monotone decreasing ..... (1)

$$c_n > \int_1^{n+1} \frac{1}{x} dx - \log n = \log \frac{n+1}{n} > 0$$

$c_n$  is bounded ..... (2)

By Weierstrass' theorem, (1) and (2), we find that  $c_n$  converges.

*Remark 1.* By putting  $\lim_n c_n = C$

$$C = \lim_n c_n \left( 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \log n \right)$$

0.577218

We call C Euler's constant.

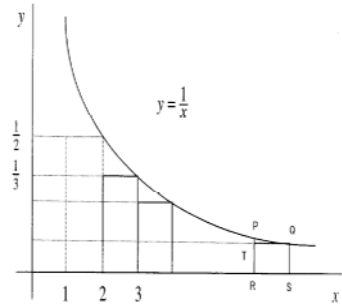


Figure 1.

*Remark 2.* By using Euler's constant, we can solve the following series.

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \log 2$$

Let  $a_n, b_n$  be as follows.

$$a_n = 1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{2n-1}$$

$$b_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \dots + \frac{1}{2n}$$

$$a_n + b_n = \log 2n + C + o \dots\dots\dots (1)$$

$$2b_n = \log n + C + o \dots\dots\dots (2)$$

$$a_n = \log 2 + \frac{1}{2} \log n + \frac{C}{2} + o$$

$$b_n = \frac{1}{2} \log n + \frac{C}{2} + o \quad ( o : \text{infinitesimal} )$$

$$a_n - b_n = \log 2$$

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \log 2$$

2. Euler's  $\Gamma$ -function

The  $\Gamma$ -function was discovered by Leonhard Euler (1707 - 1783) in 1729. This function has been studied by mathematicians to generalize  $n$  of  $n!$ . The name has been established but this name and symbol  $\Gamma(x)$  dates back to Andrei-Marie Lejandre .

$$(1) \quad \Gamma(x) = \frac{1}{x} \prod_{n=1}^{\infty} \frac{(1+\frac{1}{n})^x}{1+\frac{x}{n}} \quad (\text{Euler})$$

$$(2) \quad \Gamma(x) = \lim_n \frac{n^x n!}{x(x+1)\dots(x+n)} \quad (\text{Euler, Gauss} )$$

The following formula was introduced by Weierstrass.

$$(3) \quad \frac{1}{\Gamma(x)} = e^{Cx} x \prod_{n=1}^{\infty} (1 + \frac{x}{n}) e^{-\frac{x}{n}} \quad (\text{Weierstrass})$$

where  $C$  : Euler's constant

*Remark 3.* Above mentioned three equations, (1), (2) and (3) are equivalent. <sup>1)</sup> We can verify by easy calculation.

*Remark 4.* Equations, (2) is often used as the definition of  $\Gamma$ -function.

*Theorem 2.* Let  $C(x) = \int_0^1 e^{-t} t^{x-1} dt$ , then

$$C(x) = P(x)$$

*Proof.* 
$$C(x) = \lim_n \int_0^n \left(1 - \frac{t}{n}\right)^n t^{x-1} dt$$

By putting  $\frac{t}{n} = t$ ,

$$\begin{aligned} C(x) &= \lim_n n^x \int_0^1 (1-t)^n t^{x-1} dt \\ \int_0^1 (1-t)^n t^{x-1} dt &= \left[ \frac{1}{x} t^x (1-t)^n \right]_0^1 + \frac{n}{x} \int_0^1 (1-t)^{n-1} t^x dt \\ &= \frac{n(n-1)\dots 2 \cdot 1}{x(x+1)\dots(x+n)} \int_0^1 t^{x+n-1} dt \\ &= \frac{n!}{x(x+1)\dots(x+n)} \end{aligned}$$

$$P(x) = \lim_n \frac{n^x n!}{x(x+1)\dots(x+n)}$$

$$C(x) = P(x)$$

### 3. Euler's B- function

*Definition.* We call the following function  $B(x, y)$  B- function.

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt$$

*Remark 6.* The symbol  $B(x, y)$  dates back to Biner, J. P. M.

*Theorem 4.* 
$$B(x, y) = \frac{C(x)C(y)}{C(x+y)}$$

*Proof.* 
$$\begin{aligned} B(x, y) &= \left[ \frac{t^x}{x} (1-t)^{y-1} \right]_0^1 + \int_0^1 \frac{t^x}{x} (y-1)(1-t)^{y-2} dt \\ &= \frac{y-1}{x} \int_0^1 t^x (1-t)^{y-2} dt \\ &= \frac{y-1}{x} B(x+1, y-1) \end{aligned}$$

$$B(x, y) = \frac{y-1}{x} \frac{y-2}{x+1} \cdots \frac{1}{x+y-2} B(x+y-1, 1)$$

$$B(x+y-1, 1) = \int_0^1 t^{x+y-2} dt$$

$$= \left[ \frac{t^{x+y-1}}{x+y-1} \right]_0^1 = \frac{1}{x+y-2}$$

$$B(x, y) = \frac{(x-1)!(y-1)!}{(x+y-1)!}$$

By using (0),

$$B(x, y) = \frac{C(x)C(y)}{C(x+y)}$$

*Remark7.*

By putting  $x = y = \frac{1}{2}$ ,

$$\frac{(C(\frac{1}{2}))^2}{C(1)} = B(\frac{1}{2}, \frac{1}{2})$$

$$= \int_0^1 t^{\frac{1}{2}-1} (1-t)^{\frac{1}{2}-1} dt$$

$$= \int_0^1 \frac{1}{\sqrt{t(1-t)}} dt$$

By putting  $t = \sin^2 h$ ,

$$= 2 \int_0^{\frac{\pi}{2}} dh = \pi$$

Since  $C(1) = 1$ ,  $C(\frac{1}{2}) = \sqrt{\pi}$

$$C(\frac{3}{2}) = C(1 + \frac{1}{2}) = \frac{1}{2}C(\frac{1}{2}) = \frac{\sqrt{\pi}}{2} \dots\dots\dots(**)$$

*Theorem5.*

$$C(1+x) = \int_0^1 e^{-t} t^x dt$$

*Proof.*

$$C(x) = \int_0^1 e^{-t} t^{x-1} dt$$

By putting  $t^x = t$ , then  $dt = \frac{dt}{x t^{x-1}}$ ,  $t = t^{\frac{1}{x}}$

$$C(x) = \int_0^1 e^{-t^{\frac{1}{x}}} \frac{1}{x} dt$$

$$x C(x) = \int_0^x e^{-t/x} dt$$

Since  $x C(x) = C(1 + x)$ ,

$$C(1 + x) = \int_0^{1+x} e^{-t/x} dt$$

*Remark 8.* By putting  $x = \frac{1}{2}$ ,

$$C\left(\frac{3}{2}\right) = \int_0^{\frac{3}{2}} e^{-t^2} dt \dots\dots\dots (***)$$

By (\*\*), (\*\*\*),

$$\int_0^{\frac{3}{2}} e^{-t^2} dt = \frac{\sqrt{p}}{2} \quad (\text{de Moivre})$$

*Remark 9.* By putting  $x^2 = t$ ,

$$dx = \frac{dt}{2x} = \frac{t^{-\frac{1}{2}}}{2} dt$$

$$\begin{aligned} \int_0^{\frac{3}{2}} e^{-t^2} dt &= \frac{1}{2} \int_0^{\frac{1}{2}} e^{-t} t^{\frac{1}{2}} dt \\ &= \frac{1}{2} C\left(\frac{1}{2}\right) \\ &= C\left(\frac{3}{2}\right) = \frac{\sqrt{p}}{2} \end{aligned}$$

#### 4. Riemann's $\zeta$ - Function

This function was discovered by Euler in 1737. After Riemann pointed out that this function would do the important roll in the number theory in 1860, since the name of this function has called Riemann's zeta function.

*Definition.* We call the following function  $\zeta$  - function.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (s > 1)$$

*Theorem 6.* The  $\zeta$  - function  $\zeta(s)$  converges.

*Proof.* Letting  $S_n = \sum_{m=1}^n \frac{1}{m^s}$  ( $s > 1$ ), we find

$$\frac{1}{2^s} + \frac{1}{3^s} < \frac{2}{2^s} = \frac{1}{2^{s-1}}$$

$$\frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s} < \frac{4}{4^s} = \frac{1}{4^{s-1}}$$

.....

Let sum of the first  $2^p - 1$  terms of progression be  $S_{2^p - 1}$ ,

$$S_{2^p - 1} = \frac{1}{1^{s-1}} + \frac{1}{2^{s-1}} + \dots + \frac{1}{2^{(p-1)(s-1)}}$$

$$< \frac{1}{1 - (\frac{1}{2})^{s-1}}$$

$S_n$  is bounded.

$S_n$  is monotone increase.

By theorem of Weierstrass,  $f(s)$  converges.

*Remark.* Harmonic series converges.

$$1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} + \dots$$

$$1 + \frac{1}{2} + \frac{1}{3} + \dots > 1 + \frac{1}{2} + (\frac{1}{4} + \frac{1}{4}) + (\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}) + \dots$$

$$= 1 + \frac{1}{2} + \frac{2}{2^2} + \frac{2^2}{2^3} + \dots + \frac{2^{m-1}}{2^m} + \dots$$

$$= 1 + \frac{m}{2} + \dots$$

Let sum of the first  $2^m$  terms of progression be  $S_{2^m}$ , then  $\lim_m S_{2^m} =$

$$\lim_n S_n =$$

*Lemma 1.* ( Infinite geometric series )

$$\frac{1}{1 - x} = 1 + x + x^2 + x^3 + \dots \quad ( |x| < 1 )$$

*Theorem 7.* 
$$f(s) = \prod_p \frac{1}{1 - \frac{1}{p^s}} \quad (p: \text{prime number}, s > 1)$$

*Proof.*

$$\begin{aligned} \prod_p \frac{1}{1 - \frac{1}{p^s}} &= (1 + \frac{1}{2^s} + (\frac{1}{2^s})^2 + \dots)(1 + \frac{1}{3^s} + (\frac{1}{3^s})^2 + \dots) \\ &\quad \times (1 + \frac{1}{5^s} + (\frac{1}{5^s})^2 + \dots) \dots \\ &= 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \dots \\ &= f(s) \end{aligned}$$

*Lemma 2.* If  $0 < x < \frac{1}{2}$ , then  $\frac{1}{1-x} < 10^x$

$$e^{2x} = 1 + 2x + \frac{1}{2!}(2x)^2 + \dots > 1 + 2x$$

Since  $1 - x < \frac{1}{2}$ ,

$$e^{2x}(1-x) > (1+2x)(1-x) = 1 + x(1-2x) > 1$$

$$e^{2x} > \frac{1}{1-x}$$

Since  $1 - x < \frac{1}{2}$ , then  $e^{2x} < \frac{1}{1-x}$

Since  $e^{2x} < 10^x$ ,

$$\frac{1}{1-x} < 10^x$$

*Theorem 8.* The following infinite series diverges.

$$\sum_p \frac{1}{p} \quad (p: \text{prime number})$$

*Proof.*

$$\begin{aligned} \prod_p \frac{1}{1 - \frac{1}{p}} &= (1 + \frac{1}{2} + (\frac{1}{2})^2 + \dots)(1 + \frac{1}{3} + (\frac{1}{3})^2 + \dots) \\ &\quad \times (1 + \frac{1}{5} + (\frac{1}{5})^2 + \dots) \dots \\ &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots \end{aligned}$$

$$\prod_p \frac{1}{1 - \frac{1}{p}} = \dots\dots\dots ( )$$

By Lemma 2

$$\prod_p \frac{1}{1 - \frac{1}{p}} < \prod_p 10^{\frac{1}{p}} = 10^{\sum_p \frac{1}{p}} \dots\dots\dots ( )$$

By ( ) ( ),

$$\sum_p \frac{1}{p} =$$

*Remark 11. This theorem means that the number of prime numbers is infinit.*

### References

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