

## *A Study of Japanese Mathematics, Wasan*

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### *Introduction*

*We have to properly maintain our culture, inherit it and hand down to the next generation. This is our duty.*

*The Japanese mathematics, Wasan was discontinued in the Meiji Restoration. The Meiji administration was urged to select which mathematics, Wasan or European mathematics should be adopted in the introduction of educational system to Japan. The decision was reversed again and again, after the rough going, European mathematics was adopted in the end.*

*As a result, Wasan was extinguished. I think this decision was absolutely right but the extinction left the cause of later trouble, particularly we have no chance to teach Wasan in the mathematics class. Feeling the necessity to evaluate Wasan from the European mathematics, I began to restudy it.*

*Having been interested in the history of mathematics from a youth, I have occasionally collected materials of the history of mathematics. Fortunately, I have enough time to spare due to my retirement of teaching job.*

*I described the process of my study and opinions about mathematics. Upon reevaluating Wasan, I paid attention to fairness as follows.*

- 1) I described Wasan with present mathematics notation.*
- 2) I evaluated Wasan from the view point of modern mathematics.*

### *1 Times table, kuku, 九九*

*Kuku, nine-nine algorithm is the fundamentals of multiplication, which was implanted from China in the 8th century or so. On account of the fact that many books in the Nara or Muromachi periods, Manyoshu, Kuchizutae, Shukaisho, etc. were quoted kuku. For instance, old Japanese antholopology Manyoshu quoted kuku as like a pun as follows. <sup>18)</sup>*

*Wakakusano niitamakurawo makisomete yowoya hedatemu nikukuaranakuni.*

*(How many nights have passed, in vain, since first I lay with you, my lovely lass?) <sup>26)</sup>*

*We Japanese write it in Chinese characters as follows.*

*若草乃、新手枕乎、卷始而、夜哉将間、二八十一不在国*

*The word 八十一 in the upper sentence is literally read eighty one, which is equal to the product of nine and nine, kuku in Japanese. So, we read the word eighty one as kuku.*

*We studied kuku in the second grade of elementary school, wondering why they say it kuku, and repeatedly practiced it in chorus. I think this practice is a starting point of school education.*

*Remark: In kuku, there are devices in chanting them rhythmically from "2 times 2 is 4," "ni ni nga shi," to "9 times 9 is 81," "ku ku hachijuichi," and this order is fixed. Kuku was arranged from "9 times 9" to "2 times 2" at first, so it was called kuku by adopting the first one as its name representing the whole.*

*There is the following description in the Master Sun's Arithmetical Manuals, 孫氏算經 (c. 4th or 5th century)*

九九八十一、自相乘得幾何、答曰六千五百六十一。八九七十二、自相乘得五千一百八十四。  
..... 一二如二、自相乘得四。 一一如一、自相乘得一。

*Nine times nine is eighty one. What is the number that squares the product? It is six thousand five hundred sixty one. Eight times nine is seventy two. The square of the product is five thousand one hundred eighty four. ....One times two is two. The square of the product is four. One times one is one. The square of the product is one.*

*In consequence, they guessed that old kuku was arranged from "9 times 9" to "2 times 2." In recent years, "mokkan," narrow strips of wood on which an official message is written, unearthed in Tun huang, Juan, and other places of the northwestern provinces of China, and the order was confirmed.*

## *2 Method of Five Sparrows and Six Swallows, 五雀六燕の術 in the Nine Chapters on the Mathematical Art, 九章算術*

*The book "College Students Who can not solve Fractions" <sup>1)</sup> has become the topic of a talk for a long time ago. In these days, I hear school teachers' voices, "There are a lot of students in the class, who have lost their interest in science and mathematics and abhor them." And they say such trend "Risu banare," the students' trend to study away from science and mathematics. This trend dates back from Heian or Kamakura period (c. 9th century 14th century), for example, there is the following article in the collection of letters, Izumo Traffic, "Izumo Orai." <sup>2)</sup>*

*Lord of Izumo deplores in his letter, complaining of his subordinates making so many calculation errors, saying "Nobody is asking to know the theory of five sparrows and six swallows, but he could at least correctly calculate." I wonder what is the theory? <sup>2)</sup>*

( 1 ) *The method of five sparrows and six swallows*, 五雀六燕の術

*The nine Chapters on the Mathematical Art (hereafter Nine Chapters for brevity), 九章算術 was compiled in the Han Dynasty of China (206 B.C. 24 A.D.). There is the following problem in the chapter 8 of the book.*

*Problem 9: Now there are 5 sparrows and 6 swallows. Weigh. Sparrows are heavy and swallows are light. Exchange 1 sparrow and 1 swallow. Their weights are exactly balanced. The swallows and the sparrows together weigh 1 jin. Tell: 1 swallow and 1 sparrow. How much does each weigh?*

*Answer: Sparrow weigh  $1\frac{13}{19}$  liang. Swallow weigh  $1\frac{3}{19}$  liang.*

*Method: Solve by the Array Rule. After the exchange, each group of birds weighs 8 liang.*

<sup>4)</sup> *Remark: Conversion rule of weight units liang (両), and jin (斤)*

$$1 \text{ jin} = 16 \text{ liang}$$

*Solution 1 Comparing the weight of sparrows and swallows, we find one of swallows is equivalent to 3/4 of sparrows.*

$$5 + \frac{3}{4} \times 6 = \frac{19}{2}$$

$$\text{Sparrow: } 1(\text{jin}) \div \frac{19}{2} = \frac{2}{19} (\text{jin}) \dots\dots\text{Ans.}$$

$$\text{Swallow: } \frac{2}{19} \times \frac{3}{4} = \frac{3}{38} (\text{jin}) \dots\dots\text{Ans.}$$

*Solution 2*

$$\text{weight of 5 sparrows : weight of 6 swallows} = 5 \times 4 : 6 \times 3$$

$$= 20 : 18$$

$$= \frac{20}{38} : \frac{18}{38}$$

$$\text{weight of 5 sparrows} = 1 \times \frac{20}{38} = \frac{20}{38} (\text{jin})$$

$$\text{weight of 1 sparrow} = \frac{20}{38} \div 5 \quad \text{weight of 1 swallow} = \frac{18}{38} \div 6$$

$$= \frac{4}{38} (\text{jin}) \quad = \frac{3}{38} (\text{jin}) \dots\dots\text{Ans.}$$

$$= \frac{2}{19} (\text{jin}) \dots\dots\text{Ans.}$$

*Solution 3 Let  $x$  be the weight of a sparrow, and  $y$  be that of a swallow.*

$$\begin{cases} 5x + 6y = 1 \\ 3x = 4y \end{cases}$$

$$x = \frac{2}{19} \text{ (jin) } \dots\dots\dots\text{Ans.}$$

$$y = \frac{3}{4} \times \frac{2}{19} = \frac{3}{38} \text{ (jin) } \dots\dots\text{Ans.}$$

*Remark: Let the weight of 1 sparrow be  $x$  jin and that of 1 swallow be  $y$  jin.*

*If we transfer one of the sparrows to the swallows side,*

*sparrows side:  $5x - x$ , swallows side:  $6y + x$ .*

*Successively we transfer one of the swallows to the sparrows side,*

*sparrows side:  $5x - x + y$ , swallows side:  $6y + x - y$*

*Because both sides are equal,*

$$5x - x + y = 6y + x - y$$

$$3x = 4y$$

$$\text{In the case } x = 1, \text{ we find } y = \frac{3}{4}.$$

*Therefore, a swallow is equivalent to  $\frac{3}{4}$  of sparrows.*

### 3 Crane and Tortoise Arithmetic

*The next arithmetic problem is called "crane and tortoise arithmetic," whose name is used as a synonym for arithmetic. It is said that the problem appeared first in 1815 as follows,*

*Problem Now there are cranes and tortoises, the total is one hundred. The total number of their feet is two hundred seventy two.*

*Tell: 1 crane and 1 tortoise. What is each number?*

*Answer: The number of cranes is sixty four. The number of tortoises is thirty six. <sup>12)</sup>*

*Solution 1 If we substitute tortoises for cranes, whole the number of tortoises' feet is  $4 \times 100$ .*

*On the other hand, whole the number of cranes' feet is  $4 \times 100 - 272$ .*

*Because the number of crane's feet is two, whole the number of cranes is*

$$(4 \times 100 - 272) \div 2 = 64$$

*Solution 2* Let  $x$  be the number of cranes, and  $y$  be the number of tortoises,

$$100 - x = y \dots\dots\dots(1)$$

$$(1) \times 4: \quad 400 - 4x = 4y$$

Because whole the number of feet is 272,

$$2x + (400 - 4x) - 272 = 0$$

$$x = 64 \dots\dots\dots\text{Ans.}$$

By using (1),  $y = 36 \dots\dots\dots\text{Ans.}$

When we were elementary school children, we were used to solve such problems by the same way of solution 1. After junior high school, we solved them by the way of solution 2.

Historically speaking, simultaneous equations had been introduced into Japan in the Nara period with the Nine Chapters. In the Chapter 8, Rectangular Arrays, there is the following problem.

*Problem 1* <sup>8)</sup> Now given 3 bundles of top grade paddy, 2 bundles of medium grade paddy, and 1 bundle of low grade paddy. Yield: 39 dou of grain. 2 bundles of top grade paddy, 3 bundles of medium grade paddy, and 1 bundle of low grade paddy, yield 34 dou.

1 bundle of top grade paddy, 2 bundles of medium grade paddy, and 3 bundles of low grade paddy, yield 26 dou.

*Tell:* How much paddy does one bundle of each grade yield?

*Answer;* Top grade paddy yield  $9\frac{1}{4}$  dou per bundle; medium paddy  $4\frac{1}{4}$  dou; and low grade paddy  $2\frac{3}{4}$  dou. <sup>8)</sup>

*Remark:* Dou is one of the units of capacity in China.

*Solution* Let  $x$ ,  $y$ ,  $z$  be the capacity of top, medium, low grade of paddy respectively.

$$\begin{cases} 3x + 2y + z = 39 \dots\dots\dots (1) \\ 2x + 3y + z = 34 \dots\dots\dots (2) \\ x + 2y + 3z = 26 \dots\dots\dots (3) \end{cases}$$

$$3 \times (2): \quad 6x + 9y + 3z = 102 \dots\dots\dots (4)$$

$$(4) - (1) \times 2: \quad 5y + z = 24 \dots\dots\dots (5)$$

$$3 \times (3): \quad 3x + 6y + 9z = 78 \dots\dots\dots (6)$$

$$(6) - (1): \quad 4y + 8z = 39 \dots\dots\dots (7)$$

By (5) and (7),

$$x = 9\frac{1}{4}, \quad y = 4\frac{1}{4}, \quad z = 2\frac{3}{4}$$

#### 4 Extraction of square root in the Nine Chapters on the Mathematical Art

"Collage students who can not solve fractions" is the current topics of talk. Teaching with inappropriate method and missing a chance makes such students. We are taught the extraction in our junior high school days. Fortunately, I am able to extract a square root no matter how old I am.

This calculation was implanted into Japan from China with the Nine Chapters in the Nara period (c. 8th century). In the Chapter 4, Short width, there is the following problem. It resembles today's one in method.

Problem 12 <sup>7), 8)</sup> Now given an area 55225 square bu.

Tell: What is the side of the square?

Answer: 235 bu.

*Solution*  $200^2 < 55225 < 300^2$   
 $200 < \sqrt{55225} < 300$   
*By setting*  $55225 = (200 + x)^2$   
 $(400 + x)x = 15225$   
*Estimating*  $30 \quad x < 40$   
*By setting*  $x = 30 + y$   
 $\{400 + (30 + y)\}(30 + y) = 15225$   
 $(460 + y)y = 2325$   
*Estimating*  $5 \quad y < 6$   
 $y = 5$   
 $\sqrt{55225} = 235$

#### 5 Pythagorean Theorem

In the right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle. (The Elements proposition 47) <sup>15)</sup>

This is the famous Pythagorean Theorem or theorem of three squares, which is called so for the reason that Pythagorean School had proved the theory. There is the oldest proof still extant in the Euclidian Elements. <sup>13)</sup>

The ratio  $3 : 4 : 5$ , the special case of Pythagorean Theorem was known from the ancient times. In the ancient Egypt and Mesopotamia, this ratio was used to make right angles in the building and construction. For example, the series  $1, \frac{3}{4}, 1\frac{1}{4}$  is

described in the Rhind papyrus and the hypotenuse of rectangle in the Mesopotamian cu-reiform board, which is regarded to be calculated by using the following approximation expression. <sup>14)</sup>

$$\sqrt{a^2 + b} \quad a + \frac{b}{2a}$$

Where  $a, b$  are sides containing the right angle.

In China, Pythagorean theorem has been called Gougu Rule, 勾股. The Chou-Pei Suan-Ching, 周髀算經 is the oldest book which carries the Gougu Rule. It was written in the Han Dynasty (B.C. 2nd century) and involves the following problem.

*Problem* The distance from here to the standing pole is 60,000 li and to the sun is 80,000 li. Here is no shadow because the sun is right above us. How long is the distance from the pole to the sun? Let the distance to the pole be the base, "勾," and to the sun be altitude, 股. Square the base and the altitude. Plus both squares and extract the square root of the sum. <sup>7), 8), 10)</sup>

*Remark:* Li is one of the units of distance in China.

從髀至日下六萬里、而髀無影、從此以上至日則八萬里、若求邪至日者、以日下為勾日高為股、勾股各自乘并、而開方除之

In China, three sides of right-angled triangle are named, the shorter orthogonal side as base, 勾, the longer orthogonal side as altitude, 股, and hypotenuse as hypotenuse 弦. *Zho Shuang, 趙爽* was the first in China who proved the Pythagorean theorem in detail and wrote the proof in his book, using the figures of the square, circle and right-angled triangle, 勾股 圓方圖 as follows.

Square the base and the altitude respectively. Add both of the squares. This is equal to the square of hypotenuse.

勾股各自乘、并之為弦實、開方除之、即弦

*Proof:* In the Fig., the product of base and altitude is twice the red area. Add four times the red area to the yellow area.

This is the area of a square whose side is hypotenuse.

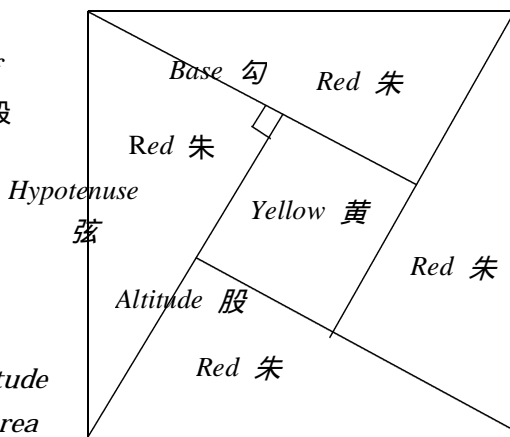


Fig. of hypotenuse, 弦 圖

Let  $a, b, c$  be base 勾, altitud 股, hypotenuse 弦 respectively.

Since 1 red area is  $\frac{1}{2}ab$ , 4 red areas are  $2ab$ , and yellow area is  $(b - a)^2$

$$2ab + (b - a)^2 = c^2$$

$$a^2 + b^2 = c^2$$

### 6 Subtraction by 105 Chinese Remainder Theorem in Jinkoki

*Jinkoki* was published in 1627 by Mitsuyoshi Yoshida, and widely used as a textbook in the private school for general people. The title of the book comes from a phrase in buddhist text, *Jinko* literally means a tremendously long time, and *Ki* means a book. <sup>24)</sup>

This book involves the following interesting problem, Subtraction by 105, 百五減算.

No. 43 This is a kind of game. Suppose there are a number of Go stones, and you should guess how many they are. Here is an example.

Suppose you are only informed that if one takes off stones 7 by 7, finally 2 stones are left. And if one takes off stones 5 by 5, finally 1 stone is left. And if one takes off stones 3 by 3, finally 2 stones are left. Knowing only these, guess the number of stones.

There are 86 stones.

Process: Multiply 2, which is the remainder when divided by 7, by 15. You get 30. Multiply 1, which is the remainder when divided by 5, by 21. You get 21. Multiply 2, which is the remainder when divided by 3, by 70. You get 140. Add these numbers. Then you get 191. If the number thus gotten exceeds 105, subtract 105 several times to get a number less than 100. In this case, you get 86. And, this is the number of stones you are asked to find. <sup>5)</sup>  $15 \times 2 + 21 \times 1 + 70 \times 2 - 105 = 86$

Remark: Using the three numbers,  $7a + 2, 5b + 1, 3c + 2$  ( $a, b, c$  : integer).

Let us make the following number,

$$P = p(5 \times 3 \times 2) + q(7 \times 3 \times 1) + r(7 \times 5 \times 2)$$

$$= 30p + 21q + 70r \quad (p, q, r : \text{integer})$$

which is divided by 7, 5, 3 and the remainders are 2, 1, 2 respectively.

P	2 (mod.7)	30p	2 (mod.7)	
		p	1 (mod.7)	p = 1
P	1 (mod.5)	21q	1 (mod.5)	
		q	1 (mod.5)	q = 1
P	2 (mod.3)	70r	2 (mod.3)	
		r	2 (mod.3)	r = 2



$$P = 15 \times 2 + 21 \times 1 + 70 \times 2$$

The number  $P$  is divided by 3, 5, 7, and their remainders are 2, 1, 2 respectively.

Moreover,  $P \equiv 86 \pmod{7 \times 5 \times 3}$

They called this theory "Subtraction by 105," because if the number is larger than 105, the number is subtracted by 105.

*Solution 1*  $x - 2 = 7l, x - 1 = 5m, x - 2 = 3n$  ( $l, m, n$  : integer)

$$x - 2 = 21p, x - 1 = 5m \quad (p: \text{integer})$$

$$x > 0 \quad p > -\frac{2}{21}, \quad 5m - 21p = 1$$

$$p = 4, \quad m = 17 \quad \text{that are well-posed.}$$

$$x = 1 + 5 \times 17 = 86 \quad \dots\dots\text{Ans.}$$

*Solution 2* The Chinese remainder theorem is described in the Master Sun's *Arithmetical Manuals*, 孫氏算經 (5th century) as follows. <sup>6)</sup>

Now, there are a lot of things and the number is unknown. When you take off things 3 by 3, 2 is left, when take off things 5 by 5, 3 is left, and when take off 7 by 7, 2 is left. Knowing only these, guess the number?

For this reason, the following theorem has been called "Chinese remainder theorem," and this theorem was proved by Gauss in the *ARITHMETICA* in 1801.

As 3, 5, 7 are relatively prime,

$$15u + 21v + 35w = 1 \quad (u, v, w : \text{integer})$$

*Particular solution:*  $u = v = 1, w = -1$

$$x = 2 \times (5 \times 3) \times 1 + 1 \times (7 \times 3) \times 1 + 2 \times (7 \times 5) \times (-1)$$

$$x \equiv -19 \pmod{105}$$

$$x = 105k - 19 \quad (\text{general solution})$$

$$\text{In the case } k = 1, \quad x = 86 \quad \dots\dots\text{Ans.}$$

## 7 Method of the celestial element, 天元術

Several Chinese mathematics books including the *Introduction to Mathematics*, 算学啓蒙 were brought back to Japan by the dispatch of Japanese troops to Korea by Hideyoshi Toyotomi (16th century).

All of the books were written about mathematics in the Sung and Yuan age of China

(10th – 14th century), especially, the solution of quadratic and cubic equations by the method of the celestial element and Horner's method, in the Introduction to Mathematics.  
 7) Remark 8) : Horner's method was independently discovered by P. Ruffini and G. Horner in 1804.

The process of the celestial element to solve equation

- (1) Set an unknown at first.
- (2) Make the equation, which satisfies the given conditions of the problem.
- (3) Make the equation by transposition, one of which sides is equal to zero.
- (4) Solve the equation.

Thus, the celestial element settled the algorithm to solve equation, and taught us the Horner's method as follows. Let us study the extraction of cubic root in the Nine Chapters in modern style as the example.

Problem 19 Short width 7)

Now given a volume 1860867 cubic chi.

Tell: What is the side of the cube?

Answer: 123

Remark: Chi is one of the units of length in China.

Solution: Let  $x$  be the length of one side of the cube,  $x^3 = 1860867$

$$x^3 - 1860867 = 0$$

Set  $x = 100x_1 \dots\dots\dots(1)$

$$1000000x_1^3 - 1860867 = 0$$

Because  $1 < x < 2$ , with estimation, we find  $x_1 = 1\dots$

By (1)  $x = 1**\dots$

Set  $x_2 = 10(x_1 - 1) \dots\dots\dots(2)$

	1000000	0	0	- 1860867
1	1000000	100000	1000000	
	1000000	1000000	1000000	- 860867
	1000000	2000000		
	1000000	2000000	3000000	
	1000000	1000000		
	1000000	3000000		

$$1000000(x_1 - 1)^3 + 3000000(x_1 - 1)^2 + 3000000(x_1 - 1) - 860867 = 0$$

By(2),  $1000x_2^3 + 30000x_2^2 + 300000x_2 - 860867 = 0$

Because  $2 < x_2 < 3$ , with estimation, we find  $x_2 = 2*...$

By (1), (2)  $x = 120$

Set  $x_3 = 10(x_2 - 2) \dots\dots\dots(3)$

	1000	30000	300000	- 860867
2		2000	64000	728000
	1000	32000	364000	- 132867
		2000	68000	
	1000	34000	432000	
		2000		
	1000	36000		

$1000(x_2 - 2)^3 + 36000(x_2 - 2)^2 + 432000(x_2 - 2) - 132867 = 0$

By (3)  $1000x_3^3 + 36000x_3^2 + 432000x_3 - 132867 = 0$

Because  $3 < x_3 < 4$ ,  $x_3 = 3$  is determined.

By (1), (2), (3)  $x = 123$

	1	360	43200	- 132867
3		3	1089	132867
	1	363	44289	0

*Remark: The celestial element is counting-rod algebra, which was born in from the late southern Sung to the beginning of the Yuan period and systematically compiled by Li Zhi, Zhu Shijie, and other scholars. But it was not long before the algebra was replaced by abacus and disappeared from China. On the other hand, the celestial element which was introduced to Korean peninsula, brought into Japan and newly developed as Wasan.*

**8 Calculator in the name of counting rods**

*They used 10 odd centimeter-long bamboo rods when they calculate or denote the numbers and they called it a counting bamboo, 算筹 in China. Since it was introduced into Japan, they have called it counting wood, 算木, hereafter refer to as counting rods.*

*We arrange the rods vertically, horizontally, or obliquely on the counting board, 算盤 as follows.*

(1) Notation

The units, hundreds, ten thousands, and so on for the odd places, were represented as follows: ○ | || ||| |||| T TT TTT TTTT | || ||| ... Vertical

0 1 2 3 4 5 6 7 8 9 100 200 300 400

The ten, thousands, hundred thousands, and so on for the even places, were represented as follows:

— = ≡ ≡≡ ≡≡≡ ⊥ ⊥ ⊥ ⊥ — = ≡ ≡≡ ... Horizontal  
 10 20 30 40 50 60 70 80 90 1000 2000 3000 4000

The blank represents zero but circle is used in script. Placing obliquely across a number, we indicate subtraction.

Example: — |||| ○ || ..... 1502      — |||| ○ || ..... 1502

Example: — ||| ..... - 15

(2) Arithmetical operation

Addition and subtraction are the same as today's method in script.

Addition      Augend ||| ≡ T ⊥ ||      43792 + 3056 = 46848  
                   Addend      ≡ ≡≡ T  
                   Sum      ||| ⊥ T ≡ T

Subtraction      Minuend ||| ≡ T ⊥ ||      43792 - 3056 = 40736  
                   Subtrahend      ≡ ≡≡ T  
                   Difference ||| T ≡ T

Multiplication      365 × 14 = 5110

1) Arrange the counting rods as in the right superposition.

Multiplicand ||| ⊥ ||||  
 Product  
 Multiplier      — |||

Hereafter denote with Arabic numerals.

2) Arrange the unit in 14 underneath the highest digit in 365.

 365  
 14
 

3) The highest digit is multiplied by 14 and arrange between 365 and 14.

 365  
 42  
 14

4) The number underneath is drawn back by one digit.

365
42
14

5) The second highest digit in 365 is multiplied by 14 and the partial products are rectified and arranged in line between 362 and 14.

365
504
14

6) The number underneath is drawn back by one digit.

365
504
14

7) The third highest digit in 365 is multiplied by 14 and the partial products are rectified and arranged in line between 362 and 14.

365
5110
14

8) Proceeding in this way, we obtain the final answer, 5110.

*Division*

$1024 \div 26$

Quotient 39, Remainder 10

1) Arrange the counting rods as in the right superposition.

Quotient  
Dividend  
Divisor

— =
= T

Hereafter denote with Arabic numerals.

2) Advance the divisor so as to the unit in the divisor under the ten's position, then 102 is divisible by the divisor.

1024
26

3) Arrange 3 for quotient in the uppermost line, we are able to subtract  $3 \times 26 = 78$  from 102.

3
244
26

4) Then divisor draw back by one digit. 244 is divisible by divisor. Arrange 9 for quotient in the uppermost line, we are able to subtract  $9 \times 26 = 234$  from 244.

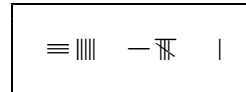
3
244
26

5) The difference 10 is the remainder, 39 in the uppermost line is the quotient.

39
10
26

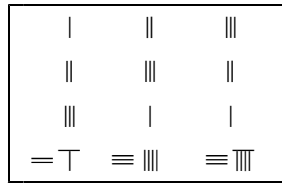
(3) Equations

1) Algebraic equation:  $x^2 - 18x + 35 = 0$



2) Simultaneous equations:

$$\begin{cases} 3x + 2y + z = 39 \\ 2x + 3y + z = 34 \\ x + 2y + 3z = 26 \end{cases}$$



9 Birth of Wasan

(1) Decipher efforts of the imported Chinese mathematics books

A lot of mathematics books including *Chou-pei Suan-ching*, 周髀算經, *Technique of Linkage*, 綴術, *Nine Chapters*, 九章算術, *Master Sun's Arithmetic Manuals*, 孫氏算經, and etc., were introduced into Japan from China during the Nara and the Heian period. The number of the books reached 175( *The list of the books in Japan*, 日本国在所目録 (889 897). But all of the books were not sufficiently digested and have lost except for the *Nine Chapters*. We call the introduction at that time the first period of introduction of mathematics. <sup>9)</sup>

The next period was started by the dispatch troops to Korea by Hideyoshi Toyotomi (1592 and 1579) and a lot of books including *Introduction to Mathematics*, 算学啓蒙, and *Yang Hui's Method of Mathematics*, 楊輝算法 were brought back to Japan. And another book from China was the *Treatise on Arithmetic*, 算法統宗. In 1658, Gentetsu Hisada published *New Edition of Introduction to Mathematics*, 算学啓蒙新編 as a reprinted edition of 算学啓蒙, since then the cipher of "the method of the celestial element" had begun by the Japanese scholars. In 1671, Kazuyuki Sawaguchi published *Kokon Sanpo-ki*, 古今算法記. He was the first person who had solved the problems by the method of the celestial element. In 1690, Katahiro Takebe published the *Great Commentary on the Introduction to Mathematics*, 算学啓蒙諺解大成, in which he had completed the cipher of 算学啓蒙.

(2) Discovery of Tenzan method, 点竄術

As we read in the previous section, we are able to calculate the algebraic equations with one variable numerical coefficients, and higher degree by the method of the celestial element. On the other hand, it always forces us mental calculation because it has no equity signs and parentheses, moreover, it is counting-rod algebra. Such inconvenience was dispelled by Takakazu Seki who designated the Tenzan method, writing algebra as

follows.

Side writing method:

1) The Chinese characters 甲, 乙, 丙, 丁..., 子, 丑, 寅, 卯, ..., etc., are used as unknowns thanks to this, the calculation of the equations with two more than variables and literal coefficients come into effect.

2) The following designated signs make the calculation process to describe on paper.

$$a \dots\dots\dots | \text{甲}$$

$$2a \dots\dots\dots || \text{甲} \text{ or } | \text{甲二}$$

$$a + b \dots\dots \left| \begin{array}{c} \text{甲} \\ \text{乙} \end{array} \right. \text{ or } | \text{甲} | \text{乙}$$

$$a - b \dots\dots\dots \left| \begin{array}{c} \text{甲} \\ \text{乙} \end{array} \right. \text{ or } | \text{甲} \backslash \text{乙}$$

$$ab \dots\dots\dots | \text{甲乙}$$

$$\frac{a}{b} \dots\dots \text{乙} | \text{甲}$$

Remark: The sign of division,  $\frac{a}{b}$  was designated after Takakazu Seki. <sup>16)</sup>

$$x^2 \dots\dots\dots \left| \begin{array}{c} \text{甲} \\ \text{巾} \end{array} \right.$$

$$x^3 \dots\dots\dots \left| \begin{array}{c} \text{甲} \\ \text{再} \\ \text{乘} \\ \text{巾} \end{array} \right. \text{ or } | \text{甲再乘巾}$$

$$\sqrt{x} \dots\dots\dots \left| \begin{array}{c} \text{甲} \\ \text{開} \\ \text{平} \\ \text{商} \end{array} \right. \text{ or } | \text{甲開平商}$$

$$\sqrt[3]{x} \dots\dots\dots \left| \begin{array}{c} \text{甲} \\ \text{開} \\ \text{立} \\ \text{商} \end{array} \right. \text{ or } | \text{甲開立商}$$

Remark: This method also has a weak point that we can not factorize the equations because it has no equity sign and parentheses. Takakazu Seki wrote the Mathematical Method for Exploring Subtle Points, 発微算法 in 1674, using the Tenzan method.

This was the epochmaking event that the Japanese mathematics departed from Chinese mathematics and took initial steps.

Generally speaking, Wasan means the Japanese mathematics subsequent to the Tenzan method. <sup>9)</sup>

### 10 Method of Joichi, 剩一, and Reiyaku, 零約

The Concise Collection of Mathematical Methods, "括用算法" Takakazu Seki's posthumous works, was published by Yoshimasa Otaka, in 1712 in which included the Joichi-jutsu and Reiyaku-jutsu, the former is the solution of Diophantin equation, and the later is the approximation method of infinite decimal to fraction.

(1) Method of Joichi, 剩一

*Problem: Now there are a lot of things and the number is not unknown. Take in 19 by 19 on the left and take off 27 by 27 on the right, finally only one thing left. How many things do you take in on the left? Answer: The total number on the left is 190.*

今有以左一十九累加之得数、以右二十七累減之、剩一、問左總數幾何。答曰、左總數百九十。  
16), 17)

*Remark: The required number is  $19x$  whose  $x$  satisfies the following indeterminate equation,*

$$19x - 27y = 1$$

*Seki calculated as follows:*  $a(27) = b(19)q_0(1) + r_0(8)$

$$b(19) = r_0(8)q_1(2) + r_1(3)$$

$$r_0(8) = r_1(3)q_2(2) + r_2(2)$$

$$r_1(3) = r_2(2)q_3(1) + r_3(1)$$

*As he has obtained  $r_3 = 1$ , he stops calculating here. Consequently*

$$p_1 = q_0q_1 + 1(\text{constant}) = 1 \cdot 2 + 1$$

$$p_2 = p_1q_2 + q_0 = 3 \cdot 2 + 1 = 7$$

$$p_3 = p_2q_3 + p_1 = 7 \cdot 1 + 3 = 10$$

*Therefore he comes to the result that  $p_3 = 10$  is the required  $x$ . In this calculation, he stops calculating, when he obtained  $r_{3m+1} = 1$  (where  $m$  is even) in  $r_{m-1} = r_mq_{m+1} + r_{m+1}$ . But if  $m$  is odd, he still continues to calculate until he comes to get*

$$r_m = r_{m+1}q_{m+2} + r_{m+2}, \quad (\text{where } r_{m+2} = 1).$$

*Solution Let us solve the indeterminate equation  $19x - 27y = 1$ .*

*Because*  $27 = 19 \times 1 + 8, 19x - (19 \times 1 + 8)y = 1$

$$19(x - y) - 8y = 1$$

*Let*  $x - y = x' \dots\dots\dots (1)$

$$19x' - 8y = 1$$

*Because*  $19 = 8 \times 2 + 3, (8 \times 2 + 3)x' - 8y = 1$

$$8(2x' - y) + 3x' = 1$$

*Let*  $2x' - y = y' \dots\dots\dots (2)$

$$8y' + 3x' = 1$$

*Because*  $8 = 3 \times 2 + 2, (3 \times 2 + 2)y' + 3x' = 1$

$$3(2y' + x') + 2y' = 1$$

*Let*  $2y' + x' = z' \dots\dots\dots (3)$

$$3z' + 2y' = 1$$



Because  $3 = 2 \times 1 + 1, \quad (2 \times 1 + 1)z' + 2y' = 1$   
 $2(z' + y') + z' = 1$

Let  $z' + y' = -p \dots \dots \dots (4)$

$-2p + z' = 1$

$z' = 1 + 2p \dots \dots \dots (5)$

By(4),(5)  $y' = 1 + 3p \dots \dots \dots (6)$

By(3),(5),(6)  $x' = 3 + 8p \dots \dots \dots (7)$

By(1),(6),(7)  $y = 7 + 19p \dots \dots \dots (8)$

By(1),(6),(7)  $x = 10 + 2p \dots \dots \dots (9)$

The general solution:  $x = 10 + 27p, \quad y = 7 + 19p \quad (p: \text{Integer})$

If  $p = 0$ , then  $x = 10, \quad y = 7$

$19x = 190$

(2) Method of Reiyaku, 零約

The method of Reiyaku is an approximation of infinite decimal to fraction.

Problem: Now given a square whose side is 1 shaku, the length of the diameter 1.41421 a little over. Approximate the diameter to fraction. <sup>17)</sup>

Remark: Shaku, "尺" is one of the units of length in Japan.

今有方一尺、斜一尺四寸一分四厘二毛一糸強、問零約之内外親疎方率各幾何。

Answer: Weak  $\frac{7}{5} < \frac{\sqrt{2}}{1} < \frac{10}{7}$ , Strong  $\frac{41}{29} < \frac{\sqrt{2}}{1} < \frac{58}{41}$

Rule: Let the series of  $a_n = \frac{b_n}{n}$  be as follows:

$$b_n = \begin{cases} b_{n-1} + 1 & \text{If } a_{n-1} > \sqrt{2} \\ b_{n-1} + 2 & \text{If } a_{n-1} < \sqrt{2} \end{cases}$$

$$a_n = \frac{1}{1}, \frac{3}{2}, \frac{4}{3}, \frac{6}{4}, \frac{7}{5}, \frac{9}{6}, \frac{10}{7}, \dots, \frac{41}{29}, \dots, \frac{58}{41}, \dots$$

Remark: Compared with the continued fraction, Seki's is very primitive form but his disciples Kaktahiro Takebe and his brother Kataaki Takebe developed the method to the continued fraction. <sup>16)</sup>

Remark: We approximate it by using continued fraction as follows:

Since  $\sqrt{2} = 1.41421\dots \quad \frac{141421}{100000}$

By using Euclidian algorithm,

141421 = 100000 × 1 + 41421	$\frac{141421}{100000} = 1 + \frac{1}{\frac{100000}{41421}}$
100000 = 41421 × 2 + 17158	$\frac{100000}{41421} = 2 + \frac{1}{\frac{17158}{41421}}$
41421 = 17158 × 2 + 7105	$\frac{41421}{17158} = 2 + \frac{1}{\frac{7105}{17158}}$
17158 = 7105 × 2 + 2948	$\frac{17158}{2948} = 5 + \frac{1}{\frac{2948}{17158}}$
7105 = 2948 × 2 + 1209	$\frac{7105}{1209} = 5 + \frac{1}{\frac{1209}{7105}}$
2948 = 1209 × 2 + 530	$\frac{2948}{530} = 5 + \frac{1}{\frac{530}{2948}}$
1209 = 530 × 2 + 149	$\frac{1209}{149} = 8 + \frac{1}{\frac{149}{1209}}$
530 = 149 × 3 + 83	$\frac{530}{83} = 6 + \frac{1}{\frac{83}{530}}$
149 = 83 × 1 + 66	.....
66 = 17 × 3 + 15	$\frac{66}{15} = 4 + \frac{1}{\frac{15}{66}}$
17 = 15 × 1 + 2	$\frac{17}{2} = 8 + \frac{1}{\frac{2}{17}}$
15 = 2 × 7 + 1	$\frac{15}{1} = 15 + \frac{1}{\frac{1}{15}}$

$$\sqrt{2} = \frac{141421}{100000}$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + 1 + \frac{1}{3} + 1 + \frac{1}{7} + \frac{1}{2}$$

$$\frac{3}{2} \left[ \frac{7}{5} \left[ \frac{17}{12} \left[ \frac{41}{29} \left[ \frac{99}{70} \left[ \frac{239}{169} \right] \right] \right] \right] \right]$$

11 Determinant as a tool which eliminates variables

In 1683, Takakazu Seki published the *Methods for Solving Concealed Problem*, 解伏題の法, in which he wrote his discovery, determinant. In the *Replacing Equations 4*, 換式第四, he wrote the concept of elimination and in the *Positive and Negative Signs 5*, 生尅第五, the theory of determinant.

(1) Replacing Equations 4, 換式第四

Problem: Given two quadratic equations, eliminate the variable,  $x^2$ .

The former equation:  $- C - B x + A x^2 = 0$

The later equation:  $- F + E x - D x^2 = 0$

In the later equation, multiply both sides by A, and in the former equation, multiply both sides by -D. Now we have two equations. The sum of the two equations is the first

equation;  $-(DC + FA) + (-DB + EA) x = 0$

Multiply both sides by  $-B$  in the former equation and by  $E$  in the former equation.

Now we have two equations. Make difference with those two equations, this is the

second equation:  $(EC + FB) - (DC + FA) x = 0$

仮如 前式、平方 後式、同

前式  $\times$  丙  $\times$  乙  $|$  甲 一式  $\times$  丁 丙  $\times$  己 甲 二式  $|$  戊 丙  $|$  己 乙

後式  $\times$  己  $|$  戊  $\times$  丁  $\times$  丁 乙  $|$  戊 甲  $\times$  丁 丙  $\times$  己 甲

以甲正遍乘後式、以丁負遍前式、相減之、得一式

以乙負遍乘後式、加一式、以戊正遍乘前式、減一式、得二式。

Describing this in a modern style, we will find as follows:

$$\begin{cases} ax^2 + bx + c = 0 & \dots\dots\dots(1) \\ dx^2 + ex + f = 0 & \dots\dots\dots(2) \end{cases}$$

(1)  $\times d$  :  $ad x^2 + bd x + cd = 0 \dots\dots\dots(3)$

(2)  $\times a$  :  $ad x^2 + ae x + af = 0 \dots\dots\dots(4)$

(4) - (3) : The first equation:  $(ae - bd) x + (af - cd) = 0 \dots\dots\dots(5)$

(2)  $\times b$  :  $bd x^2 + be x + bf = 0 \dots\dots\dots(6)$

(1)  $\times e$  :  $ae x^2 + bex + ce = 0 \dots\dots\dots(7)$

(6) - (7) :  $(bd - ae) x^2 + (bf - ce) = 0 \dots\dots\dots(8)$

From (5)  $(ae - bd) x^2 + (af - cd) x = 0 \dots\dots\dots(9)$

By (8) (9) The second equation:  $(af - cd) x + (bf - ce) = 0 \dots\dots\dots(10)$

Remark: Thus the determinant,  $ae - bd$ ,  $af - cd$ ,  $bf - ce$  plays an important role in the elimination of variables.

(2) Positive and Negative Signs 5, 生剋第五

(a) Fu, 附

Problem: In the following simultaneous equations, eliminate the variables,  $x$ ,  $x^2$ .

The first equation:  $C + Bx + Ax^2 = 0$

The second equation:  $F + Ex + Dx^2 = 0$

The third equation:  $I + Hx + Gx^2 = 0$

C E G +		BEG	AEG
F H A +		AEH	ADH
I B D +		BDH	BDG

C H D -		BDH	ADH
F B G -		AEG	BDG
I E A -		AEH	AEG

得換式、驗芟治之後、求生剋也。

仮如 一式 丙乙甲 二式 己戊丁 三式 壬辛庚

*Describing this in a modern style, we will find as follows:*

$$\begin{cases} ax^2 + bx + c = 0 & \dots\dots\dots (1) \\ dx^2 + ex + f = 0 & \dots\dots\dots (2) \\ gx^2 + hx + i = 0 & \dots\dots\dots (3) \end{cases}$$

*From (1), (2), and (3), eliminate  $x, x^2$*

$$\begin{aligned} (1) \times (-eg) : & \quad - aeg x^2 - beg x - ceg = 0 \quad \dots\dots\dots (4) \\ (1) \times dh : & \quad adh x^2 + bdh x + cdh = 0 \quad \dots\dots\dots (5) \\ (2) \times (-ah) : & \quad - adh x^2 - aeh x - afh = 0 \quad \dots\dots\dots (6) \\ (2) \times bg : & \quad bdg x^2 + beg x + bfg = 0 \quad \dots\dots\dots (7) \\ (3) \times (-bd) : & \quad - bdg x^2 - bdh x - bdi = 0 \quad \dots\dots\dots (8) \\ (3) \times ae : & \quad aeg x^2 + aeh x + aei = 0 \quad \dots\dots\dots (9) \end{aligned}$$

<i>ceg</i> -		<i>beg</i>	<i>aeg</i>
<i>afh</i> -		<i>ae</i> h	<i>adh</i>
<i>bdi</i> -		<i>bd</i> h	<i>bdg</i>

<i>cdh</i> +		<i>bd</i> h	<i>adh</i>
<i>bfg</i> +		<i>be</i> g	<i>bdg</i>
<i>aei</i> +		<i>ae</i> h	<i>aeg</i>

*Remark: In the simultaneous equations, we calculate this addition, (4) + (5) + ... + (9) in order to eliminate the variables,  $x, x^2$ .*

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - ceg - bdi - afh = 0$$

*(b) Cycle, 交式*

*In order to extend the length of cycle, we would do as follows. For instance, we add 1 to each element of cycle (123) and arrange 1 at the first position.*

$$\begin{aligned} (1 \ 1 + 1 \ 2 + 1 \ 3 + 1) &= (1 \ 2 \ 3 \ 4) \\ (1 \ 2 + 1 \ 3 + 1 \ 1 + 1) &= (1 \ 3 \ 4 \ 2) \\ (1 \ 3 + 1 \ 1 + 1 \ 2 + 1) &= (1 \ 4 \ 2 \ 3) \end{aligned}$$

*The set {(1 2 3 4), (1 3 4 2), (1 4 2 3)} is called system of equation, 換式*

從換三式、起換四式。從換四式、起換五式。逐如此。換二式換三式者、不及交式也。順逆共遞添一得次、乃式數奇者、皆順、偶者順逆相交也。

順	順	順
1	2	3

順	逆	順	逆
1	2	3	4
1	3	4	2
1	4	2	3

(順) (順) (順) (順) (順)

1	2	3	4	5
1	3	4	5	2
1	4	5	2	3
1	5	2	3	4
1	2	4	5	3
1	4	5	3	2
1	5	3	2	4
1	3	2	4	5
1	2	5	3	4
1	5	3	4	2
1	3	4	2	5
1	4	2	5	3

Remark:

$$\begin{vmatrix} 1 & 2 & 3 \\ a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} 2 & 3 & 1 \\ b & c & a \\ e & f & d \\ h & i & g \end{vmatrix} = \begin{vmatrix} 3 & 1 & 2 \\ c & a & b \\ f & d & e \\ i & g & h \end{vmatrix}$$

$$= aei + bfg + cdh - ceg - bdi - afh$$

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{vmatrix} = \begin{vmatrix} 2 & 3 & 4 & 1 \\ b & c & d & a \\ f & g & h & e \\ j & k & l & i \\ i & o & p & m \end{vmatrix} = \begin{vmatrix} 3 & 4 & 1 & 2 \\ c & d & a & b \\ g & h & e & f \\ k & l & i & j \\ o & p & m & n \end{vmatrix} = \begin{vmatrix} 4 & 1 & 2 & 3 \\ d & a & b & c \\ h & e & f & g \\ l & i & j & k \\ p & m & n & o \end{vmatrix}$$

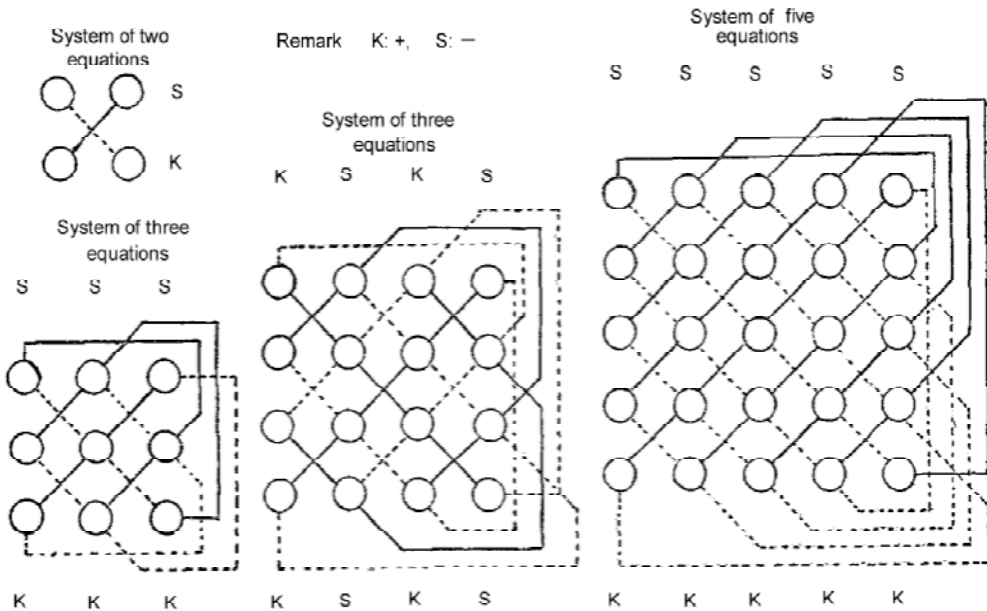


Fig. 1 <sup>16)</sup>

*Remark: The system of two and three equations is the same rule as the Sarrus' rule.*

*It has confirmed that the first discoverer of the determinants is Leibniz, who wrote it in his letters to L'Hospital in 1693. In fact, Maclaurin published his treatise, "The Maclaurin Treatise of Algebra, in 178 and the well-known Cramer's rule was published in 1750. They say that Cauchy and Jacobi were the first person who had accomplished the modern style of determinants. Today's definition of the determinant is indebted to Cauchy.* <sup>19)</sup>

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \sum \text{sgn}(i j k) a_i b_j c_k$$

w here  $\text{sgn}(i j k) = \begin{cases} 1 & A: \text{ even permutation} \\ -1 & A: \text{ odd permutation} \end{cases}, \quad A = \begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix}$

## 12 Method in the name of Circle Theory, 円理

### (1) Shigekiyo Muramatsu's pi

*The theory on the circle or sphere was called the circle theory, 円理 in Wasan. The main subject of the circle theory was the calculation of circle ratio, arc length of circle, area of circle, volume and surface area of sphere and etc. They say that Shigekiyo Muramatsu was the first who had logically calculated pi in Japan. He published the Sanso in 1663, in which he wrote about the measurement of circle ratio as follows,*

*Problem: In a square whose side is 1 shaku, inscribed about a circle, make 16-sided regular polygon inscribed in the circle. In the same way, hereafter, make regular polygon with 32 sides, 64, 128, 256, 512, 1024, 2048, 4096, 8192, 16384, 32768 and so on.*

*Remark: Shaku is one of the units of length in Japan.*

方一尺を以て八角に截り、又十六角に截り、三十二角に作り、六十四角に作り、百二十八角作り、二百五十六角に作り、五百十二角作り、千二十四角に作り、二千四十八角に作り、四千九十六角に作り八千百九十二角に作り、一万六千三百八十四角に作り、三万二千七百六十八角に截る。如图

*Given  $n$ -sided and  $2n$ -sided regular polygons inscribed in a circle, whose diameter is 1 shaku. Let the sides of  $n$ -sided and  $2n$ -sided regular polygon, be  $AB$ ,  $AC$  respectively, and the perimeter of  $n$ -sided regular polygon be  $s_n$ .*

$$AB = a_n, \quad AC = a_{2n},$$

$$CM = c_n, \quad CC' = 1$$

Since  $AM^2 = CM \cdot MC'$

$$\left(\frac{a_n}{2}\right)^2 = c_n(1 - c_n)$$

$$a_n^2 = 4c_n(1 - c_n) \dots\dots\dots(1)$$

Since  $AC^2 = AM^2 + CM^2$

$$a_{2n}^2 = \left(\frac{a_n}{2}\right)^2 + c_n^2 \dots\dots\dots(2)$$

By (1), (2)  $a_{2n}^2 = \frac{1}{2}(1 - \sqrt{1 - a_n^2})$

$$a_{2n} = \sqrt{\frac{1}{2}(1 - \sqrt{1 - a_n^2})} \quad \text{where} \quad a_4 = \frac{1}{\sqrt{2}}$$

$$s_n = n a_n$$

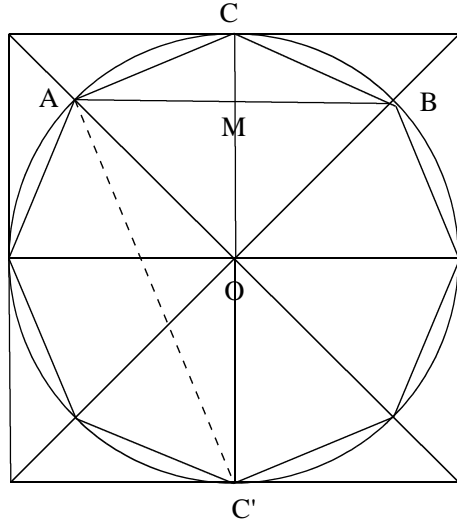


Fig. 2

Results of calculation by Mathcad 2000 (Mathsoft Inc.)<sup>1\*)</sup> are as follows:

$a_4 = 0.707106781186547$	$s_4 = 2.82842712474619$
$a_8 = 0.38268343236509$	$s_8 = 3.06146745892072$
$a_{16} = 0.195090322016128$	$s_{16} = 3.12144515225805$
$a_{32} = 0.098017140329561$	$s_{32} = 3.13654849054594$
$a_{64} = 0.049067674327418$	$s_{64} = 3.14033115695474$
$a_{128} = 0.024541228522912$	$s_{128} = 3.14127725093276$
$a_{256} = 0.012271538285719$	$s_{256} = 3.14151380114415$
$a_{512} = 6.13588464915602 \times 10^{-3}$	$s_{512} = 3.14157294036788$
$a_{1024} = 3.06795676296871 \times 10^{-3}$	$s_{1024} = 3.14158772527996$
$a_{2048} = 1.53398018628156 \times 10^{-3}$	$s_{2048} = 3.14159142150464$
$a_{4096} = 7.66990318752704 \times 10^{-4}$	$s_{4096} = 3.14159234561108$
$a_{8192} = 3.83495187566529 \times 10^{-4}$	$s_{8192} = 3.141592576545$
$a_{16384} = 1.91747597257278 \times 10^{-4}$	$s_{16384} = 3.14159263346325$
$a_{32768} = 9.58737992800168 \times 10^{-5}$	$s_{32768} = 3.14159265480759$
$a_{65536} = 4.794 \times 10^{-5}$	$s_{65536} = 3.14159264532122$
$a_{131072} = 2.397 \times 10^{-5}$	$s_{131072} = 3.14159260737572$

Remark: Result of calculation by Shigekiyo Muramatsu is as follows:

$$s_{32768} = 3.14159\ 26487\ 77698\ 86924\ 8$$

(2) Takakazu Seki's pi

Takakazu Seki calculated more accurate pi by his newly devised method, which was named "Zoyaku" method, 増約術, (a computation technique for the reduction of small decimals) as follows.

The results were written in his book, the Concise collection of Mathematical Methods, 括要算法.

Given n-sided regular polygon inscribed in a circle, whose diameter is 1 shaku. Let the perimeter of the n-sided regular polygon be  $s_n$ , we set  $a = s_{32768}$ ,  $b = s_{65536}$ ,  $c = s_{131072}$ .

$$b + \frac{(b - a)(c - b)}{(b - a) - (c - b)} = 3.14159\ 26535\ 9$$

列三万二千七百六十八角周与六万五千五百三十六角周差、以六万五千五百三十六角周与一十三万一千零七十二角周差相乘之、得数为实。列三万二千七百六十八角周与六万五千五百三十六角周差、内减六万五千五百三十六角周与一十三万一千零七十二角周差余为法。实如法而一、得数加入六万五千五百三十六角周、得三尺一寸四分一厘五毛九糸二忽六微五纤三沙五塵九埃微弱、为定周。

Remark: Results of calculation by Mathcad 2000 (Mathsoft Ltd.) are as follows: <sup>1\*)</sup>

$$\begin{aligned} a_{32768} &= 9.58737992800168 \times 10^{-5} & a &= s_{32768} = 3.14159265480759 \\ a_{65536} &= 4.794 \times 10^{-5} & b &= s_{65536} = 3.14159264532122 \\ a_{131072} &= 2.397 \times 10^{-5} & c &= s_{131072} = 3.14159260737572 \end{aligned}$$

$$b + \frac{(b - a)(c - b)}{(b - a) - (c - b)} = 3.14159\ 26579\ 6971$$

Remark: Yoshisuke Matsunaga indicated that the sequence  $a, b, c, \dots$  converges to  $d$  if the sequence  $b - a, c - b, \dots$  is geometric sequence.  $s_1 = a, s_2 = b, s_3 = c$

Estimating this sequence  $\frac{s_3 - s_2}{s_2 - s_1}, \frac{s_4 - s_3}{s_3 - s_2}, \frac{s_5 - s_4}{s_4 - s_3}, \dots$  : converges,

$$\text{we set } \frac{s_3 - s_2}{s_2 - s_1} = \frac{s_4 - s_3}{s_3 - s_2} = \frac{s_5 - s_4}{s_4 - s_3} = \dots = k$$

$$s_3 - s_2 = k(s_2 - s_1)$$

$$s_4 - s_3 = k^2(s_2 - s_1)$$

$$s_5 - s_4 = k^3(s_2 - s_1)$$

.....

$$s_n - s_{n-1} = k^{n-2}(s_2 - s_1)$$

$$\begin{aligned} &+) \\ \hline s_n - s_2 &= (k + k^2 + k^3 + \dots + k^{n-2})(s_2 - s_1) \end{aligned}$$



$$s_n - s_2 = \frac{k(1 - k^{n-2})}{1 - k}(s_2 - s_1)$$

$$k^{n-2} \quad 0 \quad s_n = s_2 + (s_2 - s_1) \frac{k}{1 - k} \dots\dots\dots(1)$$

$$\frac{s_3 - s_2}{s_2 - s_1} = k \dots\dots\dots(2)$$

By (1), (2)  $s_n = b + \frac{(b - a)(c - b)}{(b - a) - (c - b)}$

Remark: Takakazu Seki named this approximation the method of Zoyaku, 増約術.

Remark: Using the method of Reiyaku, 零約術 he showed pi as follows:

$$\frac{3}{1}, \frac{7}{2}, \frac{10}{3}, \frac{13}{4}, \frac{16}{5}, \frac{19}{6}, \frac{22}{7}, \frac{25}{8}, \frac{29}{9}, \frac{32}{10},$$

$$\frac{35}{11}, \frac{38}{12}, \frac{41}{13}, \frac{44}{14}, \frac{47}{15}, \frac{51}{16}, \frac{54}{17}, \frac{57}{18}, \frac{60}{19}, \frac{63}{20},$$

$$\frac{66}{21}, \frac{69}{22}, \frac{73}{23}, \frac{76}{24}, \frac{79}{25}, \frac{82}{26}, \frac{85}{27}, \frac{88}{28}, \frac{91}{29}, \frac{95}{30},$$

.....

$$\frac{330}{105}, \frac{333}{106}, \frac{337}{107}, \frac{340}{108}, \frac{343}{109}, \frac{346}{110}, \frac{349}{111}, \frac{352}{112}, \frac{355}{113}$$

Remark: Tsu Chung-chih's pi  $\frac{22}{7}$ ,  $\frac{355}{113}$  Liu Hui's pi  $\frac{157}{50}$  Shigeyoshi Mori's pi  $\frac{79}{25}$

(3) Katahiro Takebe's pi

Katahiro Takebe wrote more elaborate pi in his book, Mathematical Treatise on the Technique of Linkage, 綴術算経 as follows.

Let the length of  $2^n$ -sided regular polygon  $s_n$  be we set  $s_n^2 = s_n$ , and

Estimating  $\frac{s_3 - s_2}{s_2 - s_1}, \frac{s_4 - s_3}{s_3 - s_2}, \frac{s_5 - s_4}{s_4 - s_3}, \dots$  converges to  $\frac{1}{4}$ , we set as follows:

$$s_4^{(2)} = s_4^{(1)} + \frac{1}{4^2 - 1}(s_4^{(1)} - s_3^{(1)})$$

$$s_5^{(2)} = s_5^{(1)} + \frac{1}{4^2 - 1}(s_5^{(1)} - s_4^{(1)})$$

.....

$$s_{17}^{(2)} = s_{17}^{(1)} + \frac{1}{4^2 - 1}(s_{17}^{(1)} - s_{16}^{(1)})$$

$s_3^{(1)}, s_4^{(1)}, \dots, s_{17}^{(1)}$  : 1st-approximation sequence

Estimating  $\frac{s_3^{(1)} - s_2^{(1)}}{s_2^{(1)} - s_1^{(1)}}, \frac{s_4^{(1)} - s_3^{(1)}}{s_3^{(1)} - s_2^{(1)}}, \frac{s_5^{(1)} - s_4^{(1)}}{s_4^{(1)} - s_3^{(1)}}, \dots$  converges to  $(\frac{1}{4})^2$ , we set as follows:

$$s_3^{(1)} = s_3 + \frac{1}{4 - 1}(s_3 - s_2)$$

$$s_4^{(1)} = s_4 + \frac{1}{4 - 1}(s_4 - s_3)$$

.....

$$s_{17}^{(1)} = s_{17} + \frac{1}{4 - 1}(s_{17} - s_{16})$$

$s_4^{(2)}, s_5^{(2)}, \dots, s_{17}^{(2)}$  : 2nd-approximation sequence

Similarly, he calculated the next terms successively up to the 7th- approximation sequence.

$$s_{10}^{(8)} = s_{10}^{(7)} + \frac{1}{4^8 - 1}(s_{10}^{(7)} - s_9^{(7)})$$

$$\sqrt{s_{10}^{(8)}} = 3.14159 \ 26535 \ 89793 \ 23846 \ 26433 \ 83279 \ 50288 \ 41971 \text{ over}$$

Remark: For reference, I write pi correctly to fifty places decimals as follows:

3.14159 26535 89793 23846 26433 83279 50288 41971 69399 37510...

Today, we calculate pi using the following Stomer's formula and others.

$$\begin{aligned} \frac{p}{4} &= 6 \tan^{-1} \left( \frac{1}{8} \right) + 2 \tan^{-1} \left( \frac{1}{57} \right) + \tan^{-1} \left( \frac{1}{239} \right) \\ &= 6 \sum_{k=1} \frac{(-1)^{k+1} \left( \frac{1}{8} \right)^{2k-1}}{2k - 1} + 2 \sum_{k=1} \frac{(-1)^{k+1} \left( \frac{1}{57} \right)^{2k-1}}{2k - 1} + \sum_{k=1} \frac{(-1)^{k+1} \left( \frac{1}{239} \right)^{2k-1}}{2k - 1} \end{aligned}$$

(4) Toshikiyo Kamata's pi

Toshikiyo Kamata was the third generation of Takuma School in Osaka. He calculated pi by using both of the circumscribed polygons and inscribed polygons and wrote it in his book, Takuma School of Circle Theory, 宅間流円理, in 1722.

AB is a side of nth-sided regular polygon inscribed in a circle.

EF is a side of nth-sided regular polygon circumscribed in a circle.

Let AB = a, EF = b, CM = c.

In the c ase of b = 1,

$$b^2 = \frac{a^2}{1 - a^2} \dots\dots\dots(1)$$

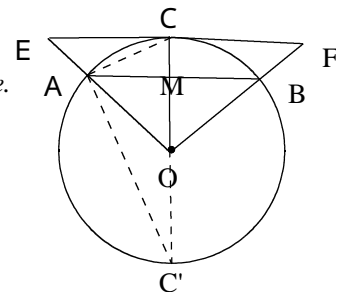


Fig. 3

Let the perimeter of 44th-sided regular polygon inscribed in a circle be  $a_{44}$ .

$$a_{44} = 3.1415\ 9265\ 3589\ 7932\ 3846\ 2643\ 3665\ 8$$

Let the perimeter of 44th-sided regular polygon circumscribed about a circle be  $b_{44}$ .

By (1),  $b_{44} = 3.1415\ 9265\ 3589\ 7932\ 3846\ 2653\ 4166\ 7$

$$3.1415\ 9265\ 3589\ 7932\ 3846\ 2653\ 4166\ 7 < p < 3.1415\ 9265\ 3589\ 7932\ 3846\ 2643\ 3665\ 8$$

Remark: At that time, there were no mathematicians who noticed the necessity both of circumscription and inscription.

Proof of (1):

	OAB	OEF	In CAC', $AM^2 = CM \cdot MC'$
	$a : b = \frac{1}{2} - c : \frac{1}{2}$		$(\frac{a}{2})^2 = c(1 - c)$
	$a = b(1 - 2c)$		$a^2 = 4c(1 - c)$
	$a^2 = b^2(1 - 2c)^2$ .....	(2)	$1 - a^2 = (1 - 2c)^2$ .....
	By (2), (3)	$b^2 = \frac{a^2}{1 - a^2}$	(3)

(5) Tomoaki Imamura's pi

Tomoaki Imamura published a work entitled the Jugai-roku, 豎亥録 in 1639. The following sentences are involved in it.

Given a circle. The diameter of the circle added to the half of the sagitta. The sum multiplied by 4 times of sagitta. Extract the square root of the product. This is the length of arc ACB.

知弧矢弦之弧式者、用知徑矢弦之徑式、知円徑、而円徑之尺數、于是知矢之尺數、半之而得尺數、俱是得尺數、于是用矢之尺數四因之而得尺數、因乘而得步數為實、用開平之式、則得尺數、是弧也。

Remark: Chinese mathematics terms, sagitta, 矢, 弧, and 弦 in the Fig.2.

sagitta, 矢 ..... arrow  $CM$  弧 ..... bow  $\overline{AB}$  弦 ..... string  $\widehat{ACB}$

Let  $AB = a$ , sagitta  $CM = c$ , diameter of circle  $CC' = d$ ,  $\widehat{ACB} = s$ ,

$$s = \sqrt{4c(d + \frac{c}{2})}$$

Remark: This approximate expression holds when  $c = \frac{d}{2}$ .

In the case where  $s = \frac{pd}{2}$  and  $c = \frac{d}{2}$ ,  $p = \sqrt{10}$

(6) Katahiro Takebe's approximation

He published the *Mathematical Treatise on the Technique of Linkage*, "綴術算経" in 1722, in which he showed the expansion of the infinitesimal arc length to an infinite power series as follows.

First of all, given a circle of 10 in diameter. Extend the arc heights 1, 2, 3, 4, 4.5, 4.9. I realized this idea is not efficient around 5.0. Mr. Seki had failed twice and I once. From the fact that the square of half an arc height 1 is 10.03 over and that of height, is  $1.0033 \cdot 10^{-1}$  over. I knew that the minutes square of half an arc, "半背巾" could solve the problem.

Therefore I took the square of half an arc height,  $10^{-5}$  and surveyed the result.

始に、径一尺にして矢一寸、矢二寸、矢三寸、矢四寸等の定背を碎約し求め続けて又矢四寸五分、矢四寸九分等の定背を求めて、その数を探るに、半円に近き者に於て據と為することを不会、故に往歳関氏孤率を造改むること再次、吾亦重て造改むこと一次、共に皆不精して其の術廢しぬ。其の矢一寸の半背巾一十寸〇三強と、矢一分の半背巾一寸〇〇三三強の数に依って、予め矢の極微なる者必ず真数の顯るべきことを察して、矢一忽半背巾の定数を求め得て其の質を探会せり

Let chord  $AB = a_1$ , chord  $AC = a_2$ , diameter  $CC' = d = 10$ ,

arc height (sagitta)  $CM = c = 10^{-5}$ ,

arc length  $\widehat{ACB} = s$

$$a_{2n}^2 = 5(10 - \sqrt{100 - a_n^2})$$

where  $a_1^2 = 4 \times 10^{-5}(10 - 10^{-5})$

$$s_n^2 = n^2 a_n^2$$

$$\left(\frac{s_5}{2}\right)^2 = \frac{1}{10^4} \times 1.0000003333335111112253969066667282347769479595875 \text{ over}$$

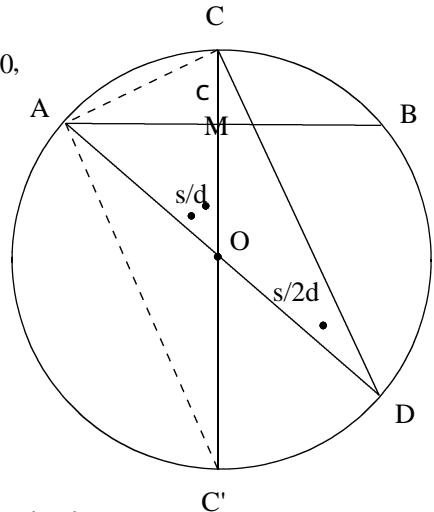


Fig. 4

Remark: He took this numerical value as a basic constant value, and called "定半背巾," and let  $s_5 = s$ .

$$\left(\frac{s}{2}\right)^2 = \frac{1}{10^4} \times 1.0000003333335111112253969066667282347769479595875$$

Estimating  $\left(\frac{s}{2}\right)^2$  is near  $\frac{1}{10^4} = \frac{1}{10^5} \times 10$ ,

Let  $cd = a_1$ ,  $a_1 = cd$  .....(1)

$$\left(\frac{s}{2}\right)^2 - a_1 = \frac{1}{10^{10}} \times 0.3333335111112253969066667282347769479595875 \text{ (the 1st difference)}$$

Estimating  $(\frac{s}{2})^2 - a_1$  is near  $\frac{1}{10^{10}} \times \frac{1}{3}$ ,

Let  $\frac{1}{10^{10}} \times \frac{1}{3} = a_2$ ,  $a_2 = \frac{1}{3} \frac{c}{d} a_1 = \frac{1}{3} (\frac{d}{c})^2 cd$  .....(2)

$(\frac{s}{2})^2 - (a_1 + a_2) = \frac{1}{10^{16}} \times 0.17777778920635733333949014436146262542$  (the 2nd difference)

Estimating  $(\frac{s}{2})^2 - (a_1 + a_2)$  is near  $\frac{1}{10^{16}} \times \frac{1}{3} \frac{8}{15}$ ,

Let  $\frac{1}{10^{16}} \times \frac{1}{3} \frac{8}{15} = a_3$ ,  $a_3 = \frac{8}{15} \frac{d}{c} a_2 = \frac{1}{3} \frac{8}{15} (\frac{d}{c})^2 cd$  .....(3)

$(\frac{s}{2})^2 - (a_1 + a_2 + a_3) = \frac{1}{10^{22}} \times 0.1142857955556171236658368484764$  (the 3rd difference)

Estimating  $(\frac{s}{2})^2 - (a_1 + a_2 + a_3)$  is near  $\frac{1}{10^{22}} \times \frac{1}{3} \frac{8}{15} \frac{9}{14}$ ,

Let  $\frac{1}{10^{22}} \times \frac{1}{3} \frac{8}{15} \frac{9}{14} = a_4$ ,  $a_4 = \frac{9}{14} \frac{d}{c} a_3 = \frac{1}{3} \frac{8}{15} \frac{9}{14} (\frac{d}{c})^3 cd$  .....(4)

$(\frac{s}{2})^2 - (a_1 + a_2 + a_3 + a_4) = \frac{1}{10^{29}} \times 0.812699028379515511341907$  (the 4th difference)

Estimating  $(\frac{s}{2})^2 - (a_1 + a_2 + a_3 + a_4)$  is near  $a_5 = \frac{1}{10^{29}} \times \frac{1}{3} \frac{8}{15} \frac{9}{14} \frac{32}{45} \frac{c}{d}$ ,

Let  $\frac{1}{10^{29}} \times \frac{1}{3} \frac{8}{15} \frac{9}{14} \frac{32}{45} \frac{c}{d} = a_5$ ,  $a_5 = \frac{32}{45} \frac{d}{c} a_4 = \frac{1}{3} \frac{8}{15} \frac{9}{14} \frac{32}{45} (\frac{d}{c})^4 cd$  ..... (5)

$(\frac{s}{2})^2 - (a_1 + a_2 + a_3 + a_4 + a_5) = \frac{1}{10^{35}} \times 0.615681102812929209$  (the 5th difference)

Estimating  $(\frac{s}{2})^2 - (a_1 + a_2 + a_3 + a_4 + a_5)$  is near  $\frac{1}{10^{29}} \times \frac{1}{3} \frac{8}{15} \frac{9}{14} \frac{32}{45} \frac{25}{33}$ ,

Let  $\frac{1}{10^{29}} \times \frac{1}{3} \frac{8}{15} \frac{9}{14} \frac{32}{45} \frac{25}{33} = a_6$ ,  $a_6 = \frac{25}{33} \frac{d}{c} a_5 = \frac{1}{3} \frac{8}{15} \frac{9}{14} \frac{32}{45} \frac{25}{33} (\frac{d}{c})^5 cd$  ... (6)

By (1), (2), (4), (5), (6), (7), (8)

$(\frac{s}{2})^2 - a_1 = \frac{1}{10^{10}} \times 0.3333335111112253969066667282347769479595875$  (the 1st difference)

By Reiyaku-jutsu,  $0.3333335111112253969066667282347769479595875 \quad \frac{1}{3}$

$(\frac{s}{2})^2 - (a_1 + a_2) = \frac{1}{10^{16}} \times 0.17777778920635733333949014436146262542$  (the 2nd difference)

By Reiyaku-jutsu, (the 2nd difference) ÷ (the 1st difference)  $0.5333 \quad 3367 \quad 6191 \quad \frac{8}{15}$

$$\left(\frac{s}{2}\right)^2 - (a_1 + a_2 + a_3 + a_4 + a_6) = \frac{1}{10^{41}} \times 0.487132313528 \text{ (the 6th difference)}$$

$$\text{(the 6th difference)} \div \text{(the 5th difference)} = 0.791209437363177936$$

$$\frac{72}{91} \text{ (By reiyaku-jutsu)}$$

We rewrite the coefficients,  $\frac{1}{3}, \frac{8}{15}, \frac{9}{14}, \frac{32}{45}, \frac{25}{33}, \frac{72}{91}$  as follows:

$$\frac{2^2}{3 \cdot 4}, \frac{4^2}{5 \cdot 6}, \frac{6^2}{7 \cdot 8}, \frac{8^2}{9 \cdot 10}, \frac{10^2}{11 \cdot 12}, \frac{12^2}{13 \cdot 14}$$

$$\left(\frac{s}{2}\right)^2 - (a_1 + a_2 + a_3) = \frac{1}{10^{22}} \times 0.1142857955556171236658368484764 \text{ (the 3rd difference)}$$

$$\text{(the 3rd difference)} \div \text{(2nd difference)} = 0.6428576000003463206203322726798884627$$

$$\frac{9}{14} \text{ (By reiyaku-jutsu)}$$

$$\left(\frac{s}{2}\right)^2 - (a_1 + a_2 + a_3 + a_4) = \frac{1}{10^{29}} \times 0.812699028379515511341907 \text{ (the 4th difference)}$$

$$\text{(the 4th difference)} \div \text{(3rd difference)} = 0.711111649832$$

$$\frac{32}{45} \text{ (By reiyaku-jutsu)}$$

$$\left(\frac{s}{2}\right)^2 - (a_1 + a_2 + a_3 + a_4 + a_5) = \frac{1}{10^{35}} \times 0.615681102812929209 \text{ (the 5th difference)}$$

$$\text{(the 5th difference)} \div \text{(the 4th difference)} = 0.757576356977$$

$$\frac{25}{33} \text{ (By reiyaku-jutsu)}$$

$$\left(\frac{s}{2}\right)^2 = cd \left\{ 1 + \frac{2^2}{3 \cdot 4} \left(\frac{c}{d}\right) + \frac{4^2}{5 \cdot 6} \left(\frac{c}{d}\right)^2 + \frac{6^2}{7 \cdot 8} \left(\frac{c}{d}\right)^3 + \frac{8^2}{9 \cdot 10} \left(\frac{c}{d}\right)^4 \right.$$

$$\left. + \frac{10^2}{11 \cdot 12} \left(\frac{c}{d}\right)^5 + \frac{12^2}{13 \cdot 14} \left(\frac{c}{d}\right)^6 + \dots \right\}$$

Remark: In the Fig. 4, since  $AC = \sqrt{cd}$ ,  $d \sin \frac{s}{2d} = AC$ ,

$$\left(\frac{s}{2}\right)^2 = d^2 \left[ \arcsin \sqrt{\frac{c}{d}} \right]^2$$

This series is the same as the following formula which was discovered by Euler: <sup>20)</sup>

$$(\arcsin \sqrt{t})^2 = t + \frac{2^2}{3 \cdot 4} t^2 + \frac{2^2 \cdot 4^2}{3 \cdot 4 \cdot 5 \cdot 6} t^3 + \frac{2^2 \cdot 4^2 \cdot 6^2}{3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} t^4 + \dots$$

(7) Takakazu Seki 's mensuration of the circle

1) Area of circle

He wrote the mensuration of the circle in his book, titled the Concise Collection of Mathematical Methods, "括要算法" as follows:

For instance, given a circle whose diameter is 7 shaku. Find the area of the circle. ...  
 Solution: As a circle's interior, regular n-th polygon inscribed in the circle is divided into congruent isosceles triangles, the total area of these triangles get closer and closer to the area of the circle and the give triangle, (Fig. 5).

仮如、有円、径七尺、問積。... 解曰、是角所極、而自中心、累圭、成此形。故周擬圭濶、半径擬圭長、求之、...

$$\begin{aligned} \text{Area of the circle} &= \frac{1}{2} \times r l \\ &= p r^2 \end{aligned}$$

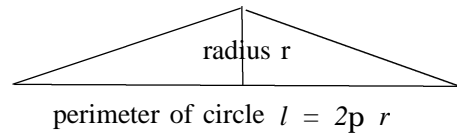


Fig. 5

2) Area of ellipse

They say that he was the first person who took up the area of ellipse in Wasan. <sup>16), 17)</sup>  
 He wrote it as follows.

For instance, given a ellipse whose major axis is 3 shaku and minor axis is 1.3 shaku. Find the volume of the ellipse.

Answer: 306.22669 cubic shaku. Solution: An ellipse is an inclined circle in shape, that is, when we cut a cylinder diagonally, the section is an ellipse. (in the Fig. 6)

仮如、有側円、長径三尺、短径一尺三寸、問積。答曰、積三百〇六寸二百二十六分寸之六十九。... 解曰、是全円欹側、而所成也。円 從上至下、斜截之則其面、即此形也。

If we write it in modern style, it will be as follows.

Let the length of the major axis be 2a, and that of minor be 2b,

since AHC ABC,

$$AH : h = 2a : 2b$$

$$AH = \frac{ah}{b} \dots\dots\dots(1)$$

Let the area of the ellipse be S,

$$p a^2 h = S \cdot AH \dots\dots\dots(2)$$

By (1), (2),  $S = p ab$

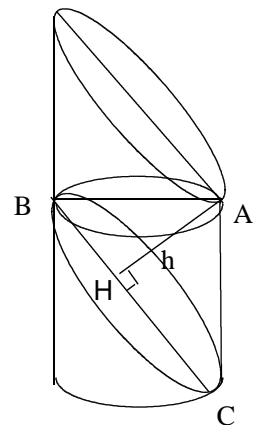


Fig. 6

3) Volume of sphere

He wrote it as follows. Given a sphere whose diameter is 1 shaku. Cut off the sphere by the planes which are parallel to the diameter AB with the width 1/50 shaku. Now, we have 50 solid bodies which he calls the square of chord, 弦巾, and add the square of upper chord to the square of lower chord. The sum is divided by 2 and multiply by the width, he calls subdivision of prism, 載積.

径一尺立円有、厚各二分、截五十片、以径矢弦術、各得弦巾。相併上下弦巾、以厚乘之、得数折半之、各得載積、...

$$(Volume\ of\ sphere) = (Diameter)^3 \times (const) \quad where \quad (const) = \frac{1}{6} \times \frac{355}{113} = \frac{355}{678}$$

Given the sphere whose diameter is  $d = 10$  sun. Cut off the sphere by plane with width  $1/n$  sun.

(i) As  $n = 50$ , Let the total of subdivision of prism be 載積  $a$ ,  $a = 666.4$

(ii) As  $n = 100$ , Let the total of subdivision of prism be 載積  $b$ ,  $b = 666.6$

(iii) As  $n = 200$ , Let the total of subdivision of prism be 載積  $c$ ,  $c = 666.65$

Using the method of Zoyaku,

$$b + \frac{(b - a)(c - b)}{(b - a) - (c - b)} = 666\frac{2}{3}$$

$$666\frac{2}{3} \times \frac{1}{4} \frac{355}{113} = \frac{1}{6} \frac{355}{113} (10)^3$$

$$= \frac{p}{6} d^3$$

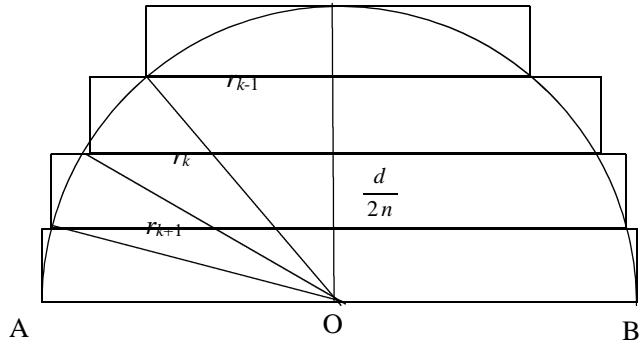


Fig. 7

Remark: Square of chord, 弦巾  $r_k^2 = d^2 \left\{ 1 - \left(1 - \frac{k}{m}\right)^2 \right\}$ ,

$$\text{Subdivision of prism, 載積 } 2 \sum_{k=0}^{m-1} \frac{r(k) + r(k+1)}{2} \frac{d}{m} \quad where \quad m = \frac{n}{2}$$

Calculation by MathCad 2000 Professional \*1)

$$\text{Square of chords } r^2_{(k)} := 10^2 \left\{ 1 - \left(1 - \frac{k}{25}\right)^2 \right\} \quad r^2_{(k)} := 10^2 \left\{ 1 - \left(1 - \frac{k}{50}\right)^2 \right\} \quad r^2_{(k)} := 10^2 \left\{ 1 - \left(1 - \frac{k}{100}\right)^2 \right\}$$

$$k := 1..25 \quad r(k) = \quad k := 1..50 \quad r(k) = \quad k := 1..100 \quad r(k) =$$

$$\text{Subdivision of prism, 載積 } a := \sum_{k=0}^{49} \frac{r(k) + r(k+1)}{2} 5^{-1} \quad 666.4,$$



$$b := \sum_{k=0}^{99} \frac{r(k) + r(k+1)}{2} 10^{-1} \quad 666.6,$$

$$c := \sum_{k=0}^{199} \frac{r(k) + r(k+1)}{2} 20^{-1} \quad 666.65$$

$$b + \frac{(b-a)(c-b)}{(b-a) - (c-b)} = 666.667$$

By the thoery of continued fraction,  $666.667 = 666\frac{2}{3}$

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1\*) *Mathcad 2000 Professional (MathSoft, Inc.)*