

The solution of cubic equation and quartic equation

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Ryoji Tatsukawa

1. Introduction

The solution was discovered in the 17th century in Italy. At that time, number was represented geometrically, that is, number, the square of number and the cubic of number were represented by segment, area and volume respectively.

As the then people had no expression by negative number, cubic equation was expressed as follows,

$$(1) \ x^3 + px = q \qquad (2) \ x^3 = px + q \qquad (3) \ x^3 + q = px$$

Scipione del Ferro (1465 – 1526) discovered the solution of (1), and Nicolo Fontana (1499 – 1557) whose another name was Tartaglia rediscovered it in 1535.

The famous scholar of Milan, Gerolamo Caldano (1501 – 1576) heard the rumor about Tartaglia's discovery, he elicited the solution of (1) by inquiring under the oath to keep it secret and extended it to (2) and (3) without delay.

Meanwhile, the solution of quartic equation was discovered by Ludovico Ferrari (around 1522 – around 1565), who was a disciple of Caldano. In spite of the oath, Caldano had to publicize the solution. Writing clearly that he got the hints to solve the equation of (1) from Tartaglia, extended the solution to (2) and (3), and the cubic equation was solved by Ferrari, he publicized the solutions in his thesis "Ars Magna" in 1545.

On the other hand, there is also criticism that Caldano had broken the promise.^{4),5)} But there is also the fact that Caldano confirmed the Ferro's discovery in his post humous works.¹⁾

2. The solution of cubic equation

Caldano's formula

Let the cubic equation be $x^3 + ax^2 + bx + c = 0$

Setting $x = y - \frac{a}{3}$, we get $y^3 + py + q = 0$

And setting $y = u + v$, we get

$$(u + v)^3 + p(u + v) + q = 0$$

$$u^3 + v^3 + (3uv + p)(u + v) + q = 0$$

$$\begin{cases} u^3 + v^3 + q = 0 \\ 3uv + p = 0 \end{cases}$$

$$\begin{cases} u^3 + v^3 = -q \\ uv = -\frac{p}{3} \end{cases}$$

$$t^2 + qt - \left(\frac{p}{3}\right)^3 = 0 \text{ (Resolvent equation) } \dots\dots\dots$$

As u^3, v^3 are solutions of the resolvent equation, we find

$$t = -\frac{q}{2} \pm \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}$$

$$u = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}, \quad v = \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}$$

$$y = u + v, \quad uy + vy^2, \quad uy^2 + vy \quad \text{where } y \text{ is a cubic root of } 1$$

Therefore, the solution of the cubic equation, $x^3 + ax^2 + bx + c = 0$ is as follows,

$$x = -\frac{a}{3} + u + v, \quad -\frac{a}{3} + uy + vy^2, \quad -\frac{a}{3} + uy^2 + vy$$

Remark: Let α, β, γ be the solutions of the cubic equation, and D be the discriminant,

$$D = \{(a - b)(b - c)(c - a)\}^2 = -4 \times 27 \left\{ \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3 \right\}$$

If we put the fundamental symmetric expressions of α, β, γ as follows,

$$v_1 = a + b + c$$

$$v_2 = ab + bc + ca$$

$$v_3 = abc$$

and because D is a homogenous expression of degree 6, every term of D is able to represent as $c v_1^{m_1} v_2^{m_2} v_3^{m_3}$ where $m_1 + 2m_2 + 3m_3 = 6$.

We find that there are seven cases with respect to (m_1, m_2, m_3) as follows.

case	(1)	(2)	(3)	(4)	(5)	(6)	(7)
m_1	0	0	1	2	3	4	6
m_2	2	3	1	2	0	1	0
m_3	2	0	1	0	1	0	0
order of x	2	3	4	4	4	5	6

Because, the order of D is 4 with respect to x , the case (6) or (7) does not happen. Therefore, we are able to write D as follows,

$$D = A_1v_3^2 + A_2v_2^3 + A_3v_1v_2v_3 + A_4v_1^2v_2^2 + A_5v_1^3v_3$$

Substituting random values for v_1, v_2, v_3 , we have the following cases.

case	(1)	(2)	(3)	(4)	(5)
	2	1	2	1	1
	2	-1	-1	1	1
	-1	0	-1	-1	0
	0	4	0	0	0
v_1	3	0	0	1	2
v_2	0	-1	-3	-1	1
v_3	-4	0	2	-1	0

$$4A_1 - 27A_5 = 0 \quad \dots\dots\dots \{ \text{by case (1)} \}$$

$$- A_2 = 4 \quad \dots\dots\dots \{ \text{by case (2)} \}$$

$$4A_1 - 27A_2 = 0 \quad \dots\dots\dots \{ \text{by case (3)} \}$$

$$A_1 - A_2 + A_3 + A_4 - A_5 = 0 \quad \dots\dots \{ \text{by case (4)} \}$$

$$A_2 + 4A_4 = 0 \quad \dots\dots\dots \{ \text{by case (5)} \}$$

$$A_1 = - 27, \quad A_2 = - 4, \quad A_3 = 18, \quad A_4 = 1, \quad A_5 = - 4$$

$$D = - 27v_3^2 - 4v_2^3 + 18v_1v_2v_3 + v_1^2v_2^2 - 4v_1^3v_3$$

$$v_1 = a + b + c = 0, \quad v_2 = ab + bc + ca = p, \quad v_3 = abc = - q$$

$$D = - 27q^2 - 4p^3 = - 4 \times 27 \left\{ \left(\frac{q}{2} \right)^2 + \left(\frac{p}{3} \right)^3 \right\}$$

Remark: In the cubic equation, $x^3 + ax^2 + bx + c = 0$,
 setting $x = y - \frac{a}{3}$, we get $y^3 + py + q = 0 \dots\dots\dots$

We can factorize $y^3 - u^3 - v^3 - 3uvy$ as follows,

$$y^3 - u^3 - v^3 - 3uvy = (y - u - v)(y - uy - vy^2)(y - uy^2 - vy) \dots\dots\dots$$

Compared with and , we find $p = -3uv$, $q = -u^3 - v^3$

$$\begin{cases} u^3 + v^3 = -q \\ uv = -\frac{p}{3} \end{cases}$$

$$t^2 + qt - \left(\frac{p}{3}\right)^3 = 0 \quad (\text{Resolvent equation})$$

As u^3, v^3 is the solution of the resolvent equation, we find

$$t = -\frac{q}{2} \pm \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}$$

$$u = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}, \quad v = \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}$$

$$x = -\frac{a}{3} + u + v, \quad -\frac{a}{3} + uy + vy^2, \quad -\frac{a}{3} + uy^2 + vy$$

3. The solution of quartic equation

Ferrari's formula

In the quartic equation, $x^4 + ax^3 + bx^2 + cx + d = 0$, if we put $x = y - \frac{a}{4}$, we get the equation, $y^4 + py^2 + qy + r = 0$, which has no the third order term.

$$y^4 = -py^2 - qy - r$$

If we add $ty^2 + \frac{t^2}{4}$ to both sides of the equation, we obtain

$$\left(y^2 + \frac{t}{2}\right)^2 = (t - p)y^2 - qy + \left(\frac{t^2}{4} - r\right) \dots\dots\dots(1)$$

When the left side of the equation is perfect square, the following equation is held, $D = (-q)^2 - 4(t - p)\left(\frac{t^2}{4} - r\right) = 0$

Therefore,

$$t^3 - pt^2 - 4rt + (4pr - q^2) = 0 \quad (\text{Resolvent equation})$$

$$\frac{t^2}{4} - r = \frac{q^2}{4(t - p)} \dots\dots\dots(2)$$

By (1) and (2), we have

$$\begin{aligned} \left(y^2 + \frac{t}{2}\right)^2 &= (t-p)y^2 - qy + \frac{q^2}{4(t-p)} \\ &= (t-p)\left(y - \frac{q}{2(t-p)}\right)^2 \\ y^2 + \frac{t}{2} &= \pm \sqrt{t-p} \left(y - \frac{q}{2(t-p)}\right) \dots\dots\dots(3) \end{aligned}$$

Let y_1, y_2, y_3, y_4 be the solutions of the quadratic equations (3), we have the solution of the quartic equation as follows,

$$x = -\frac{a}{4} + y_1, \quad -\frac{a}{4} + y_2, \quad -\frac{a}{4} + y_3, \quad -\frac{a}{4} + y_4$$

Remark

$$\begin{aligned} y_1 + y_2 &= \sqrt{t-p}, & y_1 y_2 &= \frac{t}{2} + \frac{q}{2\sqrt{t-p}} \\ y_3 + y_4 &= -\sqrt{t-p}, & y_3 y_4 &= \frac{t}{2} - \frac{q}{2\sqrt{t-p}} \\ y_1 y_2 + y_3 y_4 &= t, & y_1 + y_2 - y_3 - y_4 &= 2\sqrt{t-p} \end{aligned}$$

Therefore we can solve quadratic equation if we have following values:

$$y_1 y_2 + y_3 y_4, \quad y_1 + y_2 - y_3 - y_4$$

Euler's method

In the quartic equation, $x^4 + px^2 + qx + r = 0$, if we put $x = u + v + w$, then

$$\begin{aligned} x^2 &= u^2 + v^2 + w^2 + 2(uv + vw + wu) \\ x^4 &= (u^2 + v^2 + w^2)^2 + 4(u^2 + v^2 + w^2)(uv + vw + wu) \\ &\quad + 4(u^2v^2 + v^2w^2 + w^2u^2) + 8uvw(u + v + w) \end{aligned}$$

Substituting those expressions for the above quartic equation, we have

$$\begin{aligned} (u^2 + v^2 + w^2)^2 + 2\{2(u^2 + v^2 + w^2) + p\}(uv + vw + wu) \\ + 4(u^2v^2 + v^2w^2 + w^2u^2) + p(u^2 + v^2 + w^2) \\ + (8uvw + q)(u + v + w) + r = 0 \end{aligned}$$

$$\begin{cases} 2(u^2 + v^2 + w^2) + p = 0 \\ 8uvw + q = 0 \\ (u^2 + v^2 + w^2)^2 + 4(u^2v^2 + v^2w^2 + w^2u^2) + p(u^2 + v^2 + w^2) + r = 0 \end{cases}$$

$$\begin{cases} u^2 + v^2 + w^2 = -\frac{p}{2} \\ u^2v^2 + v^2w^2 + w^2u^2 = \frac{p^2}{16} - \frac{r}{4} \\ uvw = -\frac{q}{8} \end{cases}$$

Let u^2, v^2, w^2 be the solutions of the following equation,

$$t^3 + \frac{p}{2}t^2 + \left(\frac{p^2}{16} - \frac{r}{4}\right)t - \left(\frac{q}{8}\right)^2 = 0$$

and let t_1, t_2, t_3, t_4 be the solution of the following equation,

$$(4t)^3 + 2p(4t)^2 + (p^2 - 4r)(4t) - q^2 = 0,$$

then we have $u = \pm\sqrt{t_1}$, $v = \pm\sqrt{t_2}$, $w = \pm\sqrt{t_3}$, $uvw = -\frac{q}{8}$

$$\begin{aligned} y_1 &= \sqrt{t_1} + \sqrt{t_2} + \sqrt{t_3}, & y_2 &= \sqrt{t_1} - \sqrt{t_2} - \sqrt{t_3}, \\ y_3 &= -\sqrt{t_1} + \sqrt{t_2} - \sqrt{t_3}, & y_4 &= -\sqrt{t_1} - \sqrt{t_2} + \sqrt{t_3} \end{aligned}$$

Sample solution 1. Solve the cubic equation, $x^3 + 3x^2 + 9x + 5 = 0$

By putting $x = y - 1$, we have

$$y^3 + 6y - 2 = 0$$

$$p = 6, \quad q = -2$$

$$R = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3 = (-1)^2 + 2^3 = 9$$

$$t_1 = -\frac{q}{2} + \sqrt{R} = 1 + 3 = 4$$

$$t_2 = -\frac{q}{2} + \sqrt{R} = 1 - 3 = -2$$

$$u = \sqrt[3]{4}, \quad v = \sqrt[3]{-2} = -\sqrt[3]{2} \quad \text{hold} \quad v = -\frac{p}{3} = -2$$

$$\begin{aligned} x &= -1 + \sqrt[3]{4} - \sqrt[3]{2}, \quad -1 + \sqrt[3]{4}y - \sqrt[3]{2}y^2, \quad -1 + \sqrt[3]{4}y^2 - \sqrt[3]{2} \\ &= -1 + \sqrt[3]{4} - \sqrt[3]{2}, \quad -1 - \frac{1}{2}(\sqrt[3]{4} - \sqrt[3]{2}) \pm \frac{\sqrt{3}}{2}(\sqrt[3]{4} + \sqrt[3]{2})i \end{aligned}$$

Sample Solution 2. Solve the quartic equation, $x^4 + 4x^3 + 8x^2 + 4x + 7 = 0$

By putting $x = y - 1$, we have

$$(y - 1)^4 + 4(y - 1)^3 + 8(y - 1)^2 + 4(y - 1) + 7 = 0$$

$$y^4 + 2y^2 - 4y + 8 = 0$$

Ferrari's formula:

$$p = 2, q = -4, r = 8$$

$$\text{Resolvent equation; } t^3 - 2t^2 - 32t + 48 = 0$$

As the left side of the equation has a factor, $t - 6$ therefore, we find $t = 6$

$$\frac{t}{2} = 3, \quad t - p = 6 - 2 = 4$$

$$(y^2 + 3)^2 = 4y^2 + 4y + 1 = (2y + 1)^2$$

$$y^2 + 3 = 2y + 1, \quad -2y - 1$$

$$y = 1 \pm i, \quad -1 \pm \sqrt{3}i$$

Because $x = -1 + y$

$$x = \pm i, \quad -2 \pm \sqrt{3}i$$

Euler's method:

$$p = 2, q = -4, r = 8$$

$$t^3 + t^2 - \frac{7}{4}t - \frac{1}{4} = 0$$

$$4t^3 + 4t^2 - 7t - 1 = 0$$

As the left side of the above equation has a factor, $x - 1$, we find

$$(t - 1)(4t^2 + 8t + 1) = 0$$

$$t = 1, \quad -1 \pm \frac{\sqrt{3}}{2}$$

Because $\sqrt{t_1} \sqrt{t_2} \sqrt{t_3} = uvw = -\frac{q}{8} = \frac{1}{2}$, we find

$$\sqrt{t_1} = 1, \sqrt{t_2} = i\sqrt{1 - \frac{\sqrt{3}}{2}} = i\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right), \sqrt{t_3} = -i\sqrt{1 + \frac{\sqrt{3}}{2}} = -i\left(\frac{\sqrt{3}}{2} + \frac{1}{2}\right)$$

$$y_1 = 1 + i\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right) - i\left(\frac{\sqrt{3}}{2} + \frac{1}{2}\right) = 1 - i$$

$$y_2 = 1 - i\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right) + i\left(\frac{\sqrt{3}}{2} + \frac{1}{2}\right) = 1 + i$$

$$y_3 = -1 + i\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right) + i\left(\frac{\sqrt{3}}{2} + \frac{1}{2}\right) = -1 + \sqrt{3}i$$

$$y_4 = -1 - i\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right) - i\left(\frac{\sqrt{3}}{2} + \frac{1}{2}\right) = -1 - \sqrt{3}i$$

Because $x = -1 + y_1, -1 + y_2, -1 + y_3, -1 + y_4$, we find $x = \pm i, -2 \pm \sqrt{3}i$

Reference Books

- 代数学の歴史 ファン・デル・ベルデン著 加藤 明史訳 1994 現代数学社
B.L. van der Waerden, A History of Algebra From alkhwarizmi to Emmy Noether Springer-Verlag New York 1985
- ガロアの理論 矢ヶ部 巖 著 1976 現代数学社
- 代数学講義 高木 貞治著 1930 共立出版
- Longman Group UK Limited, Naffield Advanced Mathematics Mathematics Reader, Essex, 1994*
- 数学史 小堀 憲 著 1956 朝倉書店
- 数学の歴史 カール・ボイヤー著 加賀美 鉄雄 浦野 由有 訳 1984 朝倉書店
Carl B. Boyer, A History of Mathematics, John Wiley & Sons, inc. 1968 New York
- ガウスが切り開いた道 S・G・ギンディキン著 三浦 信夫訳 1996
シュプリンガー・フェアラーク東京
Simon G. Gindikin, Tales of Physicists and Mathematicians, Birkhauser, 1988
- わが人生の書 ギロラモ・カルダーノ著 青木 靖三 榎本 恵美子訳 1980
社会思想社 *Gerolamo Cardano, De propria vita liber, Hieromymi Cardani, Opera Omnia, Cura Caroli Sponii, Lyon, 1663*