

## 多項式の判別式 (平成26年)

定義 多項式  $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$  ( $a_n \neq 0$ ) の判別式  $\Delta(f)$  は次のように定義される.

$$\Delta(f) = (-1)^{n(n-1)/2} \frac{1}{a_n} R(f, f')$$

ただし,  $R(f, f')$  は  $f, f'$  の終結式である.

$\Delta(f) = 0 \Leftrightarrow f(x) = 0$  は重複解を持つ.

1.  $f(x) = ax^2 + bx + c$  のとき  $\Delta(f) = b^2 - 4ac$
2.  $f(x) = 4x^3 - px - q$  (楢田関数) のとき  $\Delta(f) = 16(p^3 - 27q^2)$

1.の解  $f(x) = ax^2 + bx + c$

$$f'(x) = 2ax + b$$

$f(x) = 0, f'(x) = 0$  の共通解を  $\alpha$  とする.

$$f(\alpha) = a\alpha^2 + b\alpha + c = 0$$

$$f'(\alpha) = 2a\alpha + b = 0$$

$$R(f, f') = \begin{vmatrix} a & b & c \\ 2a & b & 0 \\ 0 & 2a & b \end{vmatrix}$$

$$= \begin{vmatrix} -a & 0 & c \\ 2a & b & 0 \\ 0 & 2a & b \end{vmatrix}$$

$$= -ab^2 + 4a^2c$$

$$\therefore \Delta(f) = (-1) \frac{1}{a} (4a^2c - ab^2)$$

$$= b^2 - 4ac$$

2.の解  $f(x) = 4x^3 - px - q$

$$f'(x) = 12x^2 - p$$

$f(x) = 0, f'(x) = 0$  の共通解  $\alpha$  をとる.

$$f(\alpha) = 4\alpha^3 - p\alpha - q = 0$$

$$f'(\alpha) = 12\alpha^2 - p = 0$$

$$\begin{aligned}
R(f, f') &= \begin{vmatrix} 4 & 0 & -p & -q & 0 \\ 0 & 4 & 0 & -p & -q \\ 12 & 0 & -p & 0 & 0 \\ 0 & 12 & 0 & -p & 0 \\ 0 & 0 & 12 & 0 & -p \end{vmatrix} \\
&= \begin{vmatrix} 4 & 0 & -p & -q & 0 \\ 0 & -8 & 0 & 0 & -q \\ 12 & 0 & -p & 0 & 0 \\ 0 & 12 & 0 & -p & 0 \\ 0 & 0 & 12 & 0 & -p \end{vmatrix} \\
&= \begin{vmatrix} -8 & 0 & 0 & -q & 0 \\ 0 & -8 & 0 & 0 & -q \\ 12 & 0 & -p & 0 & 0 \\ 0 & 12 & 0 & -p & 0 \\ 0 & 0 & 12 & 0 & -p \end{vmatrix} \\
&= -8 \begin{vmatrix} -8 & 0 & 0 & -q \\ 0 & -p & 0 & 0 \\ 12 & 0 & -p & 0 \\ 0 & 12 & 0 & -p \end{vmatrix} + q \begin{vmatrix} 0 & -8 & 0 & -q \\ 12 & 0 & -p & 0 \\ 0 & 12 & 0 & 0 \\ 0 & 0 & 12 & -p \end{vmatrix} \\
&= -8 \left( -8 \begin{vmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 12 & 0 & -p \end{vmatrix} + q \begin{vmatrix} 0 & -p & 0 \\ 12 & 0 & -p \\ 0 & 12 & 0 \end{vmatrix} \right) + q \left( 8 \begin{vmatrix} 12 & -p & 0 \\ 0 & 0 & 0 \\ 0 & 12 & -p \end{vmatrix} + q \begin{vmatrix} 12 & 0 & -p \\ 0 & 12 & 0 \\ 0 & 0 & 12 \end{vmatrix} \right) \\
&= -8 (8 p^3) + q (12^3 q) \\
&= -8^2 p^3 + 12^3 q^2 \\
\therefore \Delta(f) &= (-1)^3 \frac{1}{4} (-8^2 p^3 + 12^3 q^2) = 16 (p^3 - 27q^2)
\end{aligned}$$

別解  $f(x) = 4x^2 - px - q = 0$  の根を  $\alpha_1, \alpha_2, \alpha_3$  とする.

$$4x^2 - px - q = 4(x - \alpha_1)(x - \alpha_2)(x - \alpha_3)$$

係数を比較すると,

$$\alpha_1 + \alpha_2 + \alpha_3 = 0 \quad \text{..... ①}$$

$$\alpha_1 \alpha_2 + \alpha_2 \alpha_3 + \alpha_3 \alpha_1 = -\frac{1}{4} p \quad \text{..... ②}$$

$$\alpha_1 \alpha_2 \alpha_3 = \frac{1}{4} q \quad \text{..... ③}$$

$$\Delta = 16 (\alpha_1 - \alpha_2)^2 (\alpha_2 - \alpha_3)^2 (\alpha_3 - \alpha_1)^2 \quad \text{..... ④} \quad \text{とおく.}$$

$$\text{①②③④より,} \quad \Delta = p^3 - 27q^2$$

$$\begin{aligned}
\therefore (\alpha_1 - \alpha_2)^2 &= (\alpha_1 + \alpha_2)^2 - 4\alpha_1\alpha_2 = \alpha_3^2 - 4\alpha_1\alpha_2 \\
(\alpha_2 - \alpha_3)^2 &= (\alpha_2 + \alpha_3)^2 - 4\alpha_2\alpha_3 = \alpha_1^2 - 4\alpha_2\alpha_3 \\
(\alpha_3 - \alpha_1)^2 &= (\alpha_3 + \alpha_1)^2 - 4\alpha_3\alpha_1 = \alpha_2^2 - 4\alpha_3\alpha_1 \\
\therefore (\alpha_1 - \alpha_2)^2(\alpha_2 - \alpha_3)^2(\alpha_3 - \alpha_1)^2 &= (\alpha_3^2 - 4\alpha_1\alpha_2)(\alpha_1^2 - 4\alpha_2\alpha_3)(\alpha_2^2 - 4\alpha_3\alpha_1) \\
&= (\alpha_1\alpha_2\alpha_3)^2 + 4^2\alpha_1\alpha_2\alpha_3^2\alpha_3^2 + 4^2\alpha_1^2\alpha_2\alpha_3\alpha_1^2 + 4^2\alpha_2^2\alpha_1\alpha_2^2\alpha_3 \\
&\quad - 4\alpha_1\alpha_2\alpha_1^2\alpha_2^2 - 4\alpha_2\alpha_3\alpha_2^2\alpha_3^2 - 4\alpha_3\alpha_1\alpha_3^2\alpha_1^2 \\
&\quad - 4^3(\alpha_1\alpha_2\alpha_3)^2 \\
&= (\alpha_1\alpha_2\alpha_3)^2 - 4^3(\alpha_1\alpha_2\alpha_3)^2 \\
&\quad + 4^2(\alpha_1\alpha_2\alpha_3)(\alpha_1^3 + \alpha_2^3 + \alpha_3^3) \\
&\quad - 4(\alpha_1^3\alpha_2^3 + \alpha_2^3\alpha_3^3 + \alpha_3^3\alpha_1^3) \\
&= -64(\alpha_1\alpha_2\alpha_3)^2 \\
&\quad + 16(\alpha_1\alpha_2\alpha_3)(\alpha_1^3 + \alpha_2^3 + \alpha_3^3) \\
&\quad - 4(\alpha_1^3\alpha_2^3 + \alpha_2^3\alpha_3^3 + \alpha_3^3\alpha_1^3) \dots\dots\dots ⑤
\end{aligned}$$

高校生がよく使う等式：

$$\alpha_1 + \alpha_2 + \alpha_3 = 0 \text{ のとき } \alpha_1^3 + \alpha_2^3 + \alpha_3^3 = 3\alpha_1\alpha_2\alpha_3 \dots\dots\dots ⑥$$

$$\begin{aligned}
&\alpha_1^3\alpha_2^3 + \alpha_2^3\alpha_3^3 + \alpha_3^3\alpha_1^3 - 3(\alpha_1\alpha_2)(\alpha_2\alpha_3)(\alpha_3\alpha_1) \\
&= (\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1) \{ (\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1)^2 - 3(\alpha_1\alpha_2\alpha_3)(\alpha_1 + \alpha_2 + \alpha_3) \} \\
&\dots\dots\dots ⑦
\end{aligned}$$

①②③④⑦より

$$\begin{aligned}
\alpha_1^3\alpha_2^3 + \alpha_2^3\alpha_3^3 + \alpha_3^3\alpha_1^3 &= \left(-\frac{p}{4}\right) \left[ \left(-\frac{p}{4}\right)^2 - 3\left(\frac{q}{4}\right) \times 0 \right] + 3\left(\frac{q}{4}\right)^2 \\
&= \left(-\frac{p}{4}\right)^3 + 3\left(\frac{q}{4}\right)^2 \dots\dots\dots ⑧
\end{aligned}$$

④⑤⑥⑧より

$$\begin{aligned}
\therefore \frac{\Delta}{16} &= -63\left(\frac{q}{4}\right)^2 + 16\left(\frac{q}{4}\right) \times 3\left(\frac{q}{4}\right) - 4\left[ \left(-\frac{p}{4}\right)^3 + 3\left(\frac{q}{4}\right)^2 \right] \\
&= \frac{p^3}{16} - \left(\frac{q}{4}\right)^2 \\
&= \frac{p^3 - 27q^2}{16} \\
\therefore \Delta &= p^3 - 27q^2
\end{aligned}$$

参考文献

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