A Study of Expansion into Continued Fraction August 18, 2000

1. Introduction

The ministry of education has revised its teaching guidelines every 10 years since World War II. The following contents have vanished from the course of study or have not been adopted by the ministry.

- (1) Euclidian algorithm
- (2) Continued fraction

Proof: (1) is trivial.

Let us prove (2).

Setting (a, b) = k, we have a = ka', b = kb' (a', b' : integer) $\therefore a-b = k(a'-b')$

Therefore, k is a common measure of a and b.

Conversely, letting d be a common measure of a - b and b, we find

a-b = db', b = db'' (b', b'' : integer)a = b + db' = d(b'' + b')

Therefore, a set of common measure of a and b equilibrium that of a - b and b. \therefore (a, b) = (a-b, b)

3. Euclidian algorithm

Theorem: Suppose x_0 and x_1 are given integers,

where x_0 is divided by x_1 , let k_0 be the quotient and the x_2 remainder; where x_1 is divided by x_2 , let k_1 be the quotient and the x_3 remainder; If the same operation is repeated, then we find that this algorism is as follows: $x_0 = k_0x_1 + x_2$ $(0 \le x_2 \le x_1)$

 $x_1 = k_1 x_2 + x_3 \qquad (0 \le x_3 \le x_2)$ $x_2 = k_2 x_3 + x_4 \qquad (0 \le x_4 \le x_3)$

.....

As x_n increases, x_{m-1} is exactly divided by x_m in the end.

Then, x_m is the G.C.M. of x_0 and x_1 .

Proof: By using Lemma, we find

 $= (x_1 - 3x_2, x_2)$ = $(x_1 - k_1 x_2, x_2)$ = (x_3, x_2) = (x_2, x_3) {by using Lemma (1)} = $(x_2 - x_3, x_3)$ {by using Lemma (2)}

Because $x_1 > x_2 > x_3 > \cdots \ge 0$, x_n decreases, and it becomes 0 in the end. $\therefore (x_0, x_1) = (x_m, 0) = x_m$

therefore, x_m is the G.C.M. of x_0 and x_1 .

We can formulate the Euclidian algorithm as follows:

Example 1: (39, 90) = 3 $3 \begin{vmatrix} 39 \\ 36 \\ 3 \end{vmatrix} \begin{vmatrix} 90 \\ 28 \\ 78 \\ 12 \\ 4 \\ 12 \\ 0 \end{vmatrix} = 4$ $\therefore (39, 90) = 3$

Example2: Finding the G.C.M. of $x^3 - 3x + 2$ and $x^2 - 1$,

Note
$$x^3 - 3x + 2 = (x^2 - 1) \times x - 2(x - 1)$$

 $x^2 - 1 = -2(x - 1) \times \frac{x + 1}{-2} + 0$
 $\therefore x - 1$ is the G.C.M.

4. Continued fraction

The following fraction is called a continued fraction (regular),

$$a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}}$$

and denoted by

$$a_0 + \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots$$

5. Expanding real number into continued fraction

Suppose w is a given real number, we can decompose w into two parts.

$$w = k_0 + \frac{1}{w_1}$$

In the case where w is not integer, k_0 is the greatest integer not exceeding w.

$$\therefore \quad w_1 > 1$$
$$w_1 = k_1 + \frac{1}{w_2}$$

In the case where w_1 is not integer, k_1 is the greatest integer not exceeding w_1 .

3

1

4

2

3

2

$$\therefore w_2 > 1$$

If the same operation is repeated, we have

$$w = k_0 + \frac{1}{k_1} + \frac{1}{k_2} + \dots + \frac{1}{k_{n-1}} + \frac{1}{w_n}$$

where $k_n < w_n < k_n + 1$.

Example 1:¹⁾
$$\frac{332}{87} = 3 + \frac{1}{1} + \frac{1}{4} + \frac{1}{2} + \frac{1}{3} + \frac{1}{2}$$

 $3 \begin{vmatrix} 332 \\ 261 \\ 71 \end{vmatrix} \begin{vmatrix} 87 \\ 71 \\ 16 \\ 2 \end{vmatrix} \begin{vmatrix} 14 \\ 2 \\ 14 \\ 2 \\ 16 \end{vmatrix}$
 $k_0 = k_1 = k_1 = k_1 = k_2 = k_2 = k_2 = k_2 = k_3 = k_3 = k_4 = k_5 = k_$

Example 2:
$$\sqrt{2} = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$$

$$\sqrt{2} = 1 + (\sqrt{2} - 1) = 1 + \frac{1}{2 + \frac{1}{2 + (\sqrt{2} - 1)}} = 1 + \frac{1}{2 + \frac{1}{2 + (\sqrt{2} - 1)}} = 1 + \frac{1}{\sqrt{2} + 1} = 1 + \frac{1}{2 + (\sqrt{2} - 1)}$$

$$\therefore \sqrt{2} = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$$

$$3\frac{3}{2} + \frac{7}{5} + \frac{17}{12} + \frac{41}{29} + \frac{99}{70}$$

Example 3: 1)
$$\pi = 3.1415926535...$$

= $3 + \frac{1}{7} + \frac{1}{15} + \frac{1}{1} + \frac{1}{292} + \frac{1}{1} + \frac{1}{1} + \cdots$

This is the Tsu Tschung-Tshe's pi.

Supplementary

The Numeration system of Ancient Egypt and Greece

The ancient Egyptians had no fraction numeration system except for $\frac{2}{3}$ and unit fraction, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, \cdots , but they could write small numbers as follows. They wrote $\frac{3200}{365}$ as $8\frac{2}{3}\frac{1}{10}\frac{1}{2190}$, for example.⁴⁾ Note: $\frac{3200}{365} = 8 + \frac{2}{3} + \frac{1}{10} + \frac{1}{2190}$

The ancient Greeks also wrote it in the same way; the famous Archimedes also represented $\frac{3}{4}$ as $\swarrow' \delta'$, for example.

Note:
$$\frac{3}{4} = \frac{1}{2} + \frac{1}{4}$$

 $u' \cdots \frac{1}{2} \quad y' \cdots \frac{1}{3} \quad \delta' \cdots \frac{1}{4}$

This numeration system has been handed down in the mixed fractions.

Note: We descrive $2 + \frac{1}{3}$ as a mixed fraction $2\frac{1}{3}$.

Reference books

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