

## *A Study of Expansion into Continued Fraction*

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### *1. Introduction*

*The ministry of education has revised its teaching guidelines every 10 years since World War II. The following contents have vanished from the course of study or have not been adopted by the ministry.*

- (1) *Euclidian algorithm*      (2) *Continued fraction*

*The former has vanished, though it was widely used from the ancient Greek Times, and the later has not been adopted. In this report, I will show you its teaching contents that we have taught in our class, and verify the Tsu Tschung-Tshe's  $\pi$ ,  $\frac{355}{113} < \pi < \frac{22}{7}$ . He discovered the  $\pi$ , which was written in the ancient Chinese book "隋書律歷志" without its calculus. They say that the book gave the Japanese scholars the motives to start Japanese mathematics "wasan." <sup>3)</sup>*

*2. Lemma: Let  $(a, b)$  be the G.C.M. of  $a$  and  $b$ , then we have*

$$(1) \quad (a, b) = (b, a)$$

$$(2) \quad (a, b) = (a - b, b)$$

*Proof: (1) is trivial.*

*Let us prove (2).*

*Setting  $(a, b) = k$ , we have*

$$a = ka', \quad b = kb' \quad (a', b' : \text{integer})$$

$$a - b = k(a' - b')$$

*Therefore,  $k$  is a common measure of  $a$  and  $b$ .*

*Conversely, letting  $d$  be a common measure of  $a - b$  and  $b$ , we find*

$$a - b = db', \quad b = db'' \quad (b', b'' : \text{integer})$$

$$a = b + db' = d(b'' + b')$$

Therefore, a set of common measure of  $a$  and  $b$  equals that of  $a - b$  and  $b$ .

$$(a, b) = (a - b, b)$$

### 3. Euclidian algorithm

*Theorem:* Suppose  $x_0$  and  $x_1$  are given integers,

where  $x_0$  is divided by  $x_1$ , let  $k_0$  be the quotient and the  $x_2$  remainder;

where  $x_1$  is divided by  $x_2$ , let  $k_1$  be the quotient and the  $x_3$  remainder;

If the same operation is repeated, then we find that this algorithm is as follows:

$$x_0 = k_0x_1 + x_2 \quad (0 \quad x_2 \quad x_1)$$

$$x_1 = k_1x_2 + x_3 \quad (0 \quad x_3 \quad x_2)$$

$$x_2 = k_2x_3 + x_4 \quad (0 \quad x_4 \quad x_3)$$

.....

As  $x_n$  increases,  $x_{n-1}$  is exactly divided by  $x_n$  in the end.

Then,  $x_m$  is the G.C.M. of  $x_0$  and  $x_1$ .

*Proof:* By using Lemma, we find

$$= (x_1 - 3x_2, x_2)$$

.....

$$= (x_1 - k_1x_2, x_2)$$

$$= (x_3, x_2)$$

$$= (x_2, x_3) \quad \{\text{by using Lemma (1)}\}$$

$$= (x_2 - x_3, x_3) \quad \{\text{by using Lemma (2)}\}$$

.....

Because  $x_1 > x_2 > x_3 > \dots > 0$ ,  $x_n$  decreases, and it becomes 0 in the end.

$$(x_0, x_1) = (x_m, 0) = x_m$$

therefore,  $x_m$  is the G.C.M. of  $x_0$  and  $x_1$ .

We can formulate the Euclidian algorithm as follows:

Example1:  $(39, 90) = 3$

$$\begin{array}{r|l|l|l}
 3 & 39 & 90 & 2 \\
 & \underline{36} & \underline{78} & \\
 & 3 & 12 & 4 \\
 & & \underline{12} & \\
 & & 0 & 
 \end{array}
 \qquad (39, 90) = 3$$

Example2: Finding the G.C.M. of  $x^3 - 3x + 2$  and  $x^2 - 1$ ,

$$\begin{array}{r|l|l|l}
 x & x^3 & - 3x + 2 & x^2 & - 1 & x + 1 \\
 & \underline{x^3} & \underline{- x} & \underline{x^2} & \underline{- x} & \\
 & - 2 & - 2x + 2 & & x - 1 & 1 \\
 & & & & \underline{x - 1} & \\
 & & & & 0 & 
 \end{array}$$

$x - 1$  is the G.C.M.

Note  $x^3 - 3x + 2 = (x^2 - 1) \times x - 2(x - 1)$   
 $x^2 - 1 = - 2(x - 1) \times \frac{x + 1}{- 2} + 0$   
 $x - 1$  is the G.C.M.

#### 4. Continued fraction

The following fraction is called a continued fraction(regular),

$$a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}}$$

and denoted by

$$a_0 + \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \dots$$

5. Expanding real number into continued fraction

Suppose  $w$  is a given real number, we can decompose  $w$  into two parts.

$$w = k_0 + \frac{1}{w_1}$$

In the case where  $w$  is not integer,  $k_0$  is the greatest integer not exceeding  $w$ .

$$w_1 > 1$$

$$w_1 = k_1 + \frac{1}{w_2}$$

In the case where  $w_1$  is not integer,  $k_1$  is the greatest integer not exceeding  $w_1$ .

$$w_2 > 1$$

If the same operation is repeated, we have

$$w = k_0 + \frac{1}{k_1 + \frac{1}{k_2 + \dots + \frac{1}{k_{n-1} + \frac{1}{w_n}}}}$$

where  $k_n < w_n < k_n + 1$ .

Example 1:<sup>1)</sup>  $\frac{332}{87} = 3 + \frac{1}{1 + \frac{1}{4 + \frac{1}{2 + \frac{1}{3 + \frac{1}{2}}}}$

3	332	87	1	k <sub>0</sub> = 3
4	261	71	2	k <sub>1</sub> = 1
3	64	14	2	k <sub>2</sub> = 4
3	7	2	2	k <sub>3</sub> = 2
3	6	2	2	k <sub>4</sub> = 3
3	1	0	2	k <sub>5</sub> = 2

Example 2:  $\sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2} + \dots}}$

$$\begin{aligned}
 \sqrt{2} &= 1 + (\sqrt{2} - 1) &= 1 + \frac{1}{2 + \frac{1}{2 + (\sqrt{2} - 1)}} & k_0 = 1 \\
 &= 1 + \frac{1}{\frac{1}{\sqrt{2} - 1}} && k_1 = 2 \\
 &= 1 + \frac{1}{\sqrt{2} + 1} && k_2 = 2 \\
 &= 1 + \frac{1}{2 + (\sqrt{2} - 1)} && \dots\dots
 \end{aligned}$$

$$\sqrt{2} = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots$$

3	7	17	41	99
2	5	12	29	70

Example 3: <sup>1)</sup> p = 3. 1 4 1 5 9 2 6 5 3 5 .....  

$$= 3 + \frac{1}{7} + \frac{1}{15} + \frac{1}{1} + \frac{1}{292} + \frac{1}{1} + \frac{1}{1} + \dots$$

3	311415926535	10000000000	7	$k_0 = 3$
	300000000000	9911485745		$k_1 = 7$
15	1415926535	88514255	1	$k_2 = 15$
	1327713825	88212710		$k_3 = 1$
292	88212710	301545	1	$k_4 = 292$
	88051140	161570		$k_5 = 1$
	161570	139975		

$$p = 3 + \frac{1}{7} + \frac{1}{15} + \frac{1}{1} + \frac{1}{292} + \frac{1}{1} + \frac{1}{1} + \dots$$

22	333	355
7	106	113

*This is the Tsu Tschung-Tshe's pi.*

*Supplementary*

*The Numeration system of Ancient Egypt and Greece*

The ancient Egyptians had no fraction numeration system except for  $\frac{2}{3}$  and unit fraction,  $\frac{1}{2}$ ,  $\frac{1}{3}$ ,  $\frac{1}{4}$ , ... but they could write small numbers as follows.

They wrote  $\frac{3200}{365}$  as  $8\frac{2}{3}\frac{1}{10}\frac{1}{2190}$ , for example. <sup>4)</sup>

Note:  $\frac{3200}{365} = 8 + \frac{2}{3} + \frac{1}{10} + \frac{1}{2190}$

The ancient Greeks also wrote it in the same way; the famous Archimedes also represented  $\frac{3}{4}$  as  $\text{c}' \text{d}'$  <sup>4)</sup>, for example.

Note:  $\frac{3}{4} = \frac{1}{2} + \frac{1}{4}$

$\text{c}' \dots \frac{1}{2} \quad \text{d}' \dots \frac{1}{3} \quad \text{e}' \dots \frac{1}{4}$

*This numeration system has been handed down in the mixed fractions.*

Note: We describe  $2 + \frac{1}{3}$  as a mixed fraction  $2\frac{1}{3}$ .

*Reference books*

- 1) 初等整数論講義      高木 貞治 著      1 9 3 1      共立出版
- 2) 数と式 代数入門      遠山 啓 著      1 9 8 0      講談社
- 3) 円周率の歴史      平山 諦 著      1 9 8 0      大阪教育図書
- 4) ギリシャ数学史      T・L・ヒース著平田 寛訳      1 9 5 9      共立出版

*Thomas L. Heath, A Manual of Greek Mathematics, Oxford 1931*