1. Introduction

They say that Hipparcus systematically used trigonometry, calculated chords of circle and made a chord table, which corresponds to today's trigonometric ratio table, for the first time in about 150 B.C. Unfortunately, it is not existing today, but they say, Ptolemy Ptolemaios Claudios often cited it in his writings, Almagest; however, Gerald Toomer restored it in 1973. Let us look at what kind of table they used.

Note: The Almagest is a compilation of all the study about astronomy in the ancient Greek times. It had been literally the greatest astronomical study, till heliocentric theory was advocated in the 17th century.

2. Ptolemy's cord table

First of all, he divided a quadrant into 180° equal angles around its center and its radius is divided into 120 equal parts at every 1/2°. Secondly, he calculated the length of chord AA', crd. at every central angle. Being shown in the Table, he represented the fractions of chord using sexagesimal system.

\[
\begin{array}{ccc}
\theta & \text{crd.} & \text{Average increase of chord for every 1' increase} \\
\hline
\frac{1}{2} & 0;31,25 & 0;1,2,50 \\
1 & 1;2,50 & 0;1,2,50 \\
\frac{1}{2} & 1;34,15 & 0;1,2,50 \\
2 & 2;5,40 & 0;1,2,50 \\
\frac{1}{2} & 2;37,4 & 0;1,2,48 \\
3 & 3;8,28 & 0;1,2,48 \\
\frac{1}{2} & 3;39,52 & 0;1,2,48 \\
\end{array}
\]
Note: The half length of the chord AA' corresponds to today's sine AM. This means that they used the length of the chord instead of a ratio with two sides of a right triangle.

It was the 16th century when trigonometric ratio was dealt as a right triangle. Rheticus, Georg Johannes (1514–1576) represented it as length of side of a triangle, not a ratio. A chord table had been made at every segment of radius, till the decimal notation was established by Stevin, Simon (1548–1620).

The trigonometric ratio was defined in the four quadrants, as the ratio with two sides of the right triangle, by Euler, Leonhard (1707–1783) in the 18th century, and sine notation was also established by him.

Construction of regular inscribed polygons in a circle

The following lengths of chords are able to be calculated by using inscribed regular triangle, square and hexagon.

\[
\begin{align*}
  \text{crd.}120^\circ &= \sqrt{3} \cdot r = 60\sqrt{3} \quad 103;55'23'' \\
  \text{crd.}90^\circ &= \sqrt{2} \cdot r = 60\sqrt{2} \quad 1;24'51'' \\
  \text{crd.}60^\circ &= r = 60;
\end{align*}
\]

And we can also calculate the lengths of chords for 72° and 36° by using well-known construction of regular hexagon.

- Draw a circle centered at the Point O, of which diameter is AB, and a midperpendicular CO on AB.
- With a radius CD, draw a circle centered at the point D which is a midpoint of OB. This circle meets the diameter AB at a point E.
- The line segments EO, CE are the sides of a regular decagon and a regular pentagon, respectively.

Proof:

\[
\begin{align*}
  EB \cdot EO + OD^2 &= (ED + OD)(ED - OD) + OD^2 \\
  &= ED^2 = CD^2 = CO^2 + OD^2
\end{align*}
\]

Fig. 2
This means that the segment $EB$ is divided in external and internal ratio at the point $O$. Because, the segment $OB$ is a side of the inscribed regular pentagon, then we find that the segment $EO$ is a side of inscribed hexagon in the circle $O$.

$\square$ Proposition 9 in the books 13 of Euclid's Elements $\square$

$CE^2 = CO^2 + EO^2$

The segment $CO$ is a side of regular hexagon and $EO$ is a side of regular decagon, then we find that $CE$ is a side of regular pentagon.

$\square$ Proposition 10 in the Books 13 of Euclid's Elements $\square$

**Note:** Setting $CO = r$ in the Fig.2, we find $OD = \frac{r}{2}$.

\[
CD^2 = CO^2 + OD^2 = r^2 + \left(\frac{r}{2}\right)^2 = \frac{5}{4}r^2 \quad \implies CD = \frac{\sqrt{5}}{2}r
\]

\[
EO = ED - OD = CD - OD = \frac{\sqrt{5} - 1}{2}r
\]

\[
CE^2 = EO^2 + CO^2 = \left(\frac{\sqrt{5} - 1}{2}r\right)^2 + r^2
\]

\[
CE = \frac{\sqrt{10 - 2\sqrt{5}}}{2}r
\]

We can calculate $\text{crd.36}^\circ$ and $\text{crd.72}^\circ$ by using $\square$ and $\square$.

\[
\begin{align*}
\text{crd.36}^\circ &= \frac{\sqrt{5} - 1}{2} \quad \square 60 \quad 37;4'55'', \quad \text{crd.72}^\circ &= \frac{\sqrt{10 - 2\sqrt{5}}}{2} \quad \square 60 \quad 70;32'3''
\end{align*}
\]

\[
(3) \quad (\text{crd.a})^2 + \{\text{crd.(180}^\circ - \text{a)}\}^2 = (\text{crd.180}^\circ)^2
\]

**Proof:** Setting $AB = 1$, we get

\[
\begin{align*}
BC &= \text{crd.}, \\
AC &= \text{crd.180}^\circ - \square\square
\end{align*}
\]

\[
CB^2 + CA^2 = AB^2
\]

\[
\square \quad (\text{crd.})^2 + \{\text{crd.(180}^\circ - \square\}^2
\]

\[
= (\text{crd.180}^\circ)^2
\]

$\square$ Fig.3 $\square$
Note: Using this formula, we can calculate crd.108° and crd.44°.
\[
\text{crd.108°}^2 + \{\text{crd.}(180° - 108°)\}^2 = \text{crd.180°}^2
\]
\[
\text{crd.108°}^2 = \text{crd.180°}^2 - \text{crd.72°}^2
\]
\[
= (2r)^2 - \left(\frac{\sqrt{10 - 2\sqrt{5}}}{2}r\right)^2 = \frac{6 + 2\sqrt{5}}{4} r^2
\]
\[
\text{crd.108°} = \frac{\sqrt{5} + 1}{2} r = \frac{\sqrt{5} + 1}{2} \times 60
\]
\[
= 30 \left(\sqrt{5} + 1\right) \times 97:455''
\]

\[
\text{crd.144°}^2 + \{\text{crd.}(180° - 144°)\}^2 = \text{crd.180°}^2
\]
\[
\text{crd.144°}^2 = \text{crd.180°}^2 - \text{crd.36°}^2
\]
\[
= (2r)^2 - \left(\frac{\sqrt{3} - 1}{2}r\right)^2 = \frac{10 + 2\sqrt{5}}{4} r^2
\]
\[
\text{crd.144°} = \frac{\sqrt{10 + 2\sqrt{5}}}{2} r = \frac{\sqrt{10 + 2\sqrt{5}}}{2} \times 60
\]
\[
= 114:736''
\]

Lemma: For the inscribing quadrangle ABCD in a circle, we have
\[AB \cdot CD + BC \cdot AD = AC \cdot BD.\] \[\text{Ptolemy's theorem}\]

Proof:
Take a point E on the segment BD so that \[\angle DAC = \angle EAB.\]
\[\angle ACD = \angle ABE\]
\[\angle ADC = \angle ABE\]
\[AB : BE = AC : CD\]
\[AB \cdot CD = AC \cdot BE \ldots \ldots \]
\[\angle BAC = \angle EDA,\]
\[\angle ACB = \angle ADB\]
\[\angle ABC = \angle AED\]
\[BC : AC = ED : AD\]
\[AD \cdot BC = AC \cdot ED \ldots \ldots \]
\[\angle + \angle \quad AB \cdot CD + AD \cdot BC\]
\[= AC \cdot (BE + ED)\]
\[= AC \cdot BD\]
Theorem

Theorem 1: When chord $AB$ and $BD$ meet at an end point $B$ of diameter $BC$, if $BC = 1$, $\overline{AOC} = \theta$ and $\overline{DOC} = \phi$, the following addition theorem is established:

\[
\text{crd.}180^\circ \cdot \text{crd.}(\theta - \phi) = \text{crd.} \theta \cdot \text{crd.}(180^\circ - \phi) - \text{crd.}(180^\circ - \phi) \cdot \text{crd.} \phi
\]

Proof: By using Ptolemy's theorem, we have

\[
AD \cdot BC = AC \cdot BD - AB \cdot DC.
\]

$AD = \text{crd.}(\theta - \phi)$, $BC = \text{crd.}180^\circ$, $AC = \text{crd.} \theta$, $BD = \text{crd.}(180^\circ - \phi)$, $DC = \text{crd.} \phi$

Then, we get

\[
\text{crd.}180^\circ \cdot \text{crd.}(\theta - \phi) = \text{crd.} \theta \cdot \text{crd.}(180^\circ - \phi) - \text{crd.}(180^\circ - \phi) \cdot \text{crd.} \phi.
\]

Note: This chord addition theorem corresponds to the following sine addition theorem:

\[
sin(a - b) = \sin a \cos b - \cos a \sin b
\]

Proof:

$AB = \cos a$, $AC = \sin a$

$BD = \cos b$, $DC = \sin b$

By using Ptolemy's theorem, we have

\[
AD \cdot BC = AC \cdot BD - AB \cdot DC.
\]

In $\triangle AED$,

$\triangle AED = \triangle ABD = \theta - \phi$

$\triangle ADE = \frac{1}{2} \cdot \overline{R}$

$\triangle AD = \sin (a - b)$

By using $\triangle$ and $\triangle$, we get

\[
\sin(a - b) = \sin a \cos b - \cos a \sin b.
\]
Note: If we take the point $D$ on arc $BEC$, we get the following formula:

$$\sin(a + b) = \sin a \cos b + \cos a \sin b.$$ 

Note: We can calculate $\text{crd} 12^\circ$.

$$\text{crd} 180^\circ \cdot \text{crd} (180^\circ - 60^\circ) = \text{crd} 72^\circ \cdot \text{crd} (180^\circ - 60^\circ) \cdot \text{crd} (180^\circ - 72^\circ) \cdot \text{crd} 60^\circ$$

$$2r \cdot \text{crd} 12^\circ = \text{crd} 72^\circ \cdot \text{crd} 120^\circ - \text{crd} 108^\circ \cdot \text{crd} 60^\circ$$

$$= \frac{\sqrt{10 - 2\sqrt{5}}}{2} r \cdot \sqrt{3} r - \frac{\sqrt{5} + 1}{2} r \cdot r$$

$$\begin{align*}
\text{crd} 12^\circ &= \frac{\sqrt{30 - 6\sqrt{5}} - \sqrt{5} - 1}{4} r \\
&= \frac{\sqrt{30 - 6\sqrt{5}} - \sqrt{5} - 1}{4} \times 60 \\
&= (\sqrt{30 - 6\sqrt{5}} - \sqrt{5} - 1) \times 15 \\
&= 12;32'36''
\end{align*}$$

Theorem 2:

Setting $\square AOB = \square$ and $\square DOA = \square$ in the Fig.7, we have the following formula:

$$\text{crd} 180^\circ \cdot \text{crd} (180^\circ - BD) = \text{crd} (180^\circ - AB) \cdot \text{crd} (180^\circ - AD) - \text{crd} AB \cdot \text{crd} AD.$$ 

Proof:

AE = 2r, CD = crd $\{180^\circ - (\square + \square)\}$,
AC = crd $(180^\circ - \square)$,
DE = crd $(180^\circ - \square)$,
AD = crd $\square$, CE = crd $\angle$.

By using Ptolemy's theorem, we get

$$AE \cdot CD = AC \cdot DE + AD \cdot CE.$$ 

$$\text{crd} 180^\circ \cdot \text{crd} \{180^\circ - (a + b)\}$$

$$= \text{crd} a \cdot \text{crd} (180^\circ - b) - \text{crd} b \cdot \text{crd} a$$

Note:

$$\cos(a + b) = \cos a \cos b - \sin a \sin b$$

[Fig.7]
Proof: Setting $\angle ACB = a$ and $\angle DCA = b$ in the Fig.8, we get this formula.

$\begin{align*}
AB &= CE \\
AC &= \cos a, \quad AB = \sin a \\
DE &= \cos b, \quad AD = \sin b \\
AE &= 1, \quad CD = \cos(a + b)
\end{align*}$

By using Ptolemy’s theorem, we get

$AE \cdot CD = AC \cdot DE - AD \cdot CE.$

$\cot(a + b) = \cot a \cos b - \sin a \sin b$

Theorem 3:

$(\text{crd.} \frac{a}{2})^2 = \frac{1}{2} (\text{crd.}180^\circ) \cdot \{\text{crd.}180^\circ - \text{crd.}(180^\circ - a)\}$

Proof: Setting $\angle COB = a$ in the Fig.9, we get this formula. The bisector of $\angle COB$ meets the circle $O$ at a point $D$ and a point $E$ satisfies $AC = AE$.

So we find that $CD = DE = DB$.

Let $F$ be a foot of the perpendicular that we drop to $AB$ from the point $D$.

$EF = FB$

$EB = AB - AE = AB - AC$

$\angle DOB \parallel \angle DEF$

$OB \cdot EB = DB^2$ ................. $\square$

By using $\square$ and $\square$ we have

$DB^2 = \frac{1}{2} AB(AB - AC)$

$DB = \text{crd.} \frac{a}{2}, \quad AB = \text{crd.}180^\circ$

$AC = \text{crd.}(180^\circ - a)$

$\square$Fig.9 $\square$
\( \text{crd.} \frac{a}{2} = \frac{1}{2} (\text{crd.} 180 \frac{a}{2} - \text{crd.} 180 \frac{b}{2} - \text{crd.} a) \)

**Note:** When \( \left\langle \text{CAB} \right\rangle = \left\langle \text{a} \right\rangle \), theorem 3 corresponds to the following formula for half angle.

\[
\sin^2 \frac{a}{2} = \frac{1}{2} (1 - \cos a)
\]

We can calculate the following chords by using this formula.

\[
\begin{align*}
(\text{crd.} 6 \frac{a}{2})^2 &= (\text{crd.} 180 \frac{a}{2})(\text{crd.} 180 \frac{a}{2} - \text{crd.} 12 \frac{a}{2}) = r(2r - \text{crd.} 168 \frac{a}{2}) \\
(\text{crd.} 168 \frac{a}{2})^2 &= (\text{crd.} 180 \frac{a}{2})^2 - (\text{crd.} 12 \frac{a}{2})^2 \\
&= \frac{64 - \left(\sqrt{30 - 6\sqrt{5}} - \sqrt{5} - 1\right)^2}{16} r^2 \\
\text{crd.} 168 \frac{a}{2} &= \frac{r}{4} \sqrt{64 - \left(\sqrt{30 - 6\sqrt{5}} - \sqrt{5} - 1\right)^2} \\
\text{crd.} 6 \frac{a}{2} &= \frac{r}{2} \sqrt{8 - \sqrt{64 - \left(\sqrt{30 - 6\sqrt{5}} - \sqrt{5} - 1\right)^2}} \\
&= 6;16'49"
\end{align*}
\]

Similarly, we can calculate the following chords.

\[
\begin{align*}
\text{crd.} 3 \frac{a}{2} &= \frac{\sqrt{2}}{2} r \sqrt{4 - \sqrt{8 + \sqrt{64 - \left(\sqrt{30 - 6\sqrt{5}} - \sqrt{5} - 1\right)^2}}} \\
&= 3;8'28"
\end{align*}
\]

\[
\begin{align*}
\text{crd.} 1 \frac{a}{2} &= \frac{\sqrt{2}}{2} r \sqrt{4 - \sqrt{2} \sqrt{4 + \sqrt{8 + \sqrt{64 - \left(\sqrt{30 - 6\sqrt{5}} - \sqrt{5} - 1\right)^2}}} \\
&= 1;34'14"
\end{align*}
\]

\[
\begin{align*}
\text{crd.} 3 \frac{a}{4} &= \sqrt{2} \frac{r}{2} \sqrt{2 + \frac{\sqrt{2}}{2} \sqrt{4 + \sqrt{8 + \sqrt{64 - \left(\sqrt{30 - 6\sqrt{5}} - \sqrt{5} - 1\right)^2}}} \\
&= 0;47'7"
\end{align*}
\]

**6 Interpolation formula**

\[
\frac{\text{crd.} b}{\text{crd.} a} < \frac{b}{a} \quad \text{for} \quad 0 < a < b < \frac{p}{2}
\]

**Proof:** Let \( E \) be the point at which the bisector of the angle \( ABC \) meets the side \( AC \).
we find \( AD = CD \).
\( AB : BC = AE : EC \)
\( AE < EC \)

Let \( DF \) be a perpendicular to \( AC \), then we find,
\( AD > ED > FD \).

The circle which has a point \( D \) as a center and a radius \( DE \), meets the line \( AD \) at the point \( G \) and the extension of \( DF \) at the point \( H \).

\( EF : AE = \square EDF : \square ADE \)
\( \text{(sector}EDH) : (\text{sector}GDE) \)
\( = \square EDH : \square GDE \)
\( EF : AE < \square EDF : \square ADE \)

\( AF : AE < \square ADF : \square ADE \)
\( \text{componendo} \)
\( \text{Fig.10} \)

\( AC : AE < \square ADC : \square ADE \) \( \text{(multiplying both sides by 2)} \)
\( EC : AE < \square EDC : \square ADE \) \( \text{(dividendo)} \)
\( \square BC : AB < \square EDC : \square ADE = \frac{BC}{AB} \)
\( \square \text{crd}.b : \text{crd}.a < b : a \)

Note: We can calculate \( \text{crd}.1^\circ \) as follows:

\[
\begin{align*}
\frac{3}{4} & < 1 \frac{2}{3} < \frac{3}{2} \quad \square \quad \frac{\text{crd}.1^\circ}{\text{crd}.\frac{3}{4}} < \frac{1}{3}, \\
\frac{2}{3} \text{crd}.\frac{3}{2} & < \text{crd}.1^\circ < \frac{4}{3} \text{crd}.\frac{3}{4} \quad \square \quad \text{crd}.\frac{3}{2} \quad \text{crd}.1^\circ < \frac{3}{2} \quad \text{crd}.1^\circ < \frac{3}{2}
\end{align*}
\]

\( 1;2'50" < \text{crd}.1^\circ < 1;2'50\frac{2}{3}" \)

\( \text{crd}.1^\circ \quad \text{or} \quad 1;2'50" \)
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10 The chord Table in Ancient Greek Times