

*A Short History of Pi Before and After the Time of Archimedes*

*February 8, 2001*

*Ryoji Tatsukawa*

*1. Introduction*

*It was in about 200 B.C. that Archimedes calculated pi by using perimeters of polygons which circumscribed out or inscribed in a circle. He got pi, which is correct two places of decimals by using regular 96 sided-polygons.*

*A lot of mathematicians had calculated pi by using this geometrical method. Of all the mathematicians, Ludolf van Ceulen (1540 1610) had devoted his life to calculating pi, which is correct 35 decimal places by using regular  $2^{62}$ - polygons.*

*Though, this geometrical method had completed by Willbrod Snellius (1581 1626) and Christian Huygens (1629 1695), the calculation of pi had not developed till the late 17th century.*

*After the binomial theorem and the differential and integral calculus had discovered, James Gregory (1638 1675) discovered the series for pi, in 1671, which we human beings had long looked forward to. It took almost 2,000 years to find it since Archimedes' pi.*

*Let us study pi in the pre- and post- Archimedes' times.*

*2. Ancient Egyptian Pi*

*An English antique collector, A. Henry Rhind purchased the ancient Egyptian papyruses in Luxor in 1858. They had been transcribed from the older ones by priest Ahmes from B.C. 1849 to B.C. 1801. They are called "Rhind Mathematics*

Papyrus."

A mathematics historian, August Eisenlohr read and commented on it in his report in 1877. The following sentence is one of them; finding the area of a circle having a diameter of nine "roothen" (unit of length), let us square the difference of the diameter and its one fifth.

If we calculate it in a modern style, it goes as follows; letting  $d$  be a diameter of a circle, we find

$$(The\ area\ of\ a\ circle) = (d - \frac{1}{9}d)^2 = (\frac{8}{9})^2d^2$$

Because, the area of a circle is  $p\left(\frac{d}{2}\right)^2$ , we get  $(\frac{8}{9}d)^2 = p(\frac{d}{2})^2$

$$p = (\frac{16}{9})^2 \quad 3.16$$

The ancient Egyptian had no idea of pi, but this means that they calculated the area of a circle by using 3.16 as pi. How did they get the fraction  $\frac{8}{9}$ ?

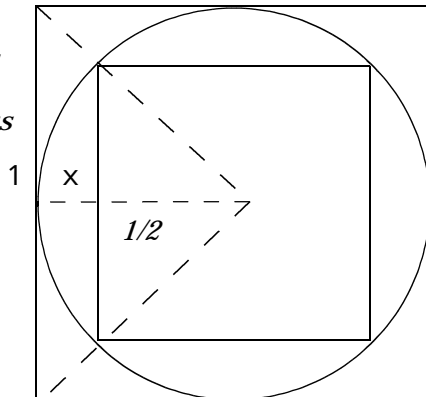
There are three views about this as follows;

(1) We consider two squares circumscribed about or inscribed in a circle with a radius of 1.

(A side of square inscribed in a circle)

$$= \frac{1}{\sqrt{2}} \quad \frac{7}{9}$$

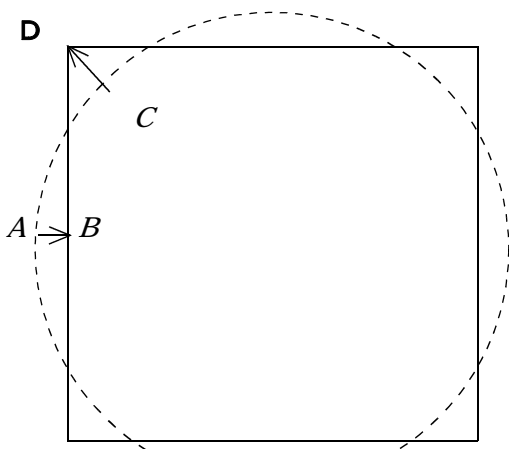
Fraction  $\frac{8}{9}$  is a mean of 1 and  $\frac{7}{9}$ .



(Fig.1)

(2) Deforming a circle to a square without changing its area and corresponding points A, C to points B, D respectively, we can measure the arc length  $\widehat{AC}$  and length BD, and let them as follows;  $\widehat{AC} = \frac{8}{9} BD$

Letting  $d$  be a side of square and  $r$  be a radius of a circle, we find



$$\frac{2pr}{8} = \frac{8}{9} \times \frac{d}{2} \quad \frac{d}{r} = \frac{9p}{16} \dots\dots$$

$$p r^2 = d^2 \quad p = \left(\frac{d}{r}\right)^2 \dots\dots$$

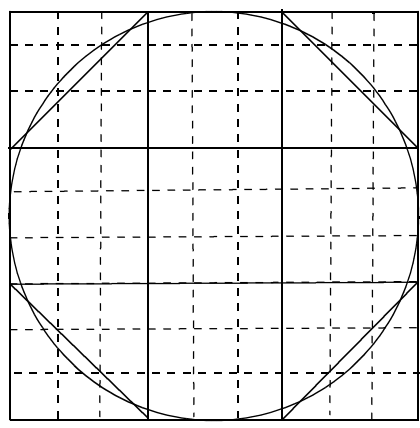
Using and , we get

$$p = \left(\frac{16}{9}\right)^2$$

(Fig.2)

(3) We can find the area of a circle by counting square units.

Consider a circle inscribing in a square whose side is 9. We count nearly 64 square units contained in a circle. This means five squares whose areas are 9, four rectangular equilateral triangles whose areas are 9/2 and a square whose area is 1, are 64 in total.

$$64 \times \left(\frac{1}{9}\right)^2 = \left(\frac{8}{9}\right)^2$$


(Fig.3)

### 3. Expansion of Pi (From the end of the 17th century to the beginning of the 18th century)

(1) Gregory's series (Leibnitz's series)

The following series was discovered by James Gregory ( 1638 1675) in 1671 and rediscovered by Leibnitz in 1673. This series was the one which human beings

had long looked forward to.

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

*Proof:*  $\int \frac{dx}{1+x^2} = \tan^{-1} x \quad (|x| < 1)$

$$\int \frac{dx}{1+x^2} = \int (1 - x^2 + x^4 - x^6 + \dots) dx$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

Setting  $x = 1$ , we have  $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$

*Remark: The convergence of this series is slow but the next three series are so fast that they are practical.*

(2) Euler's series

$$\frac{\pi}{4} = \left\{ \frac{1}{2} - \frac{1}{3} \left(\frac{1}{2}\right)^3 + \frac{1}{5} \left(\frac{1}{2}\right)^5 - \dots \right\} + \left\{ \frac{1}{3} - \frac{1}{3} \left(\frac{1}{3}\right)^3 + \frac{1}{5} \left(\frac{1}{3}\right)^5 - \dots \right\}$$

*This series was discovered by Leonhard Euler (1707 - 1783) in 1737.*

*Proof:*  $\tan^{-1} a + \tan^{-1} b = \tan^{-1} \frac{a+b}{1-ab} \dots$

Putting  $a = \frac{1}{2}$ ,  $b = \frac{1}{3}$  in , we find

$$\tan^{-1} 1 = \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3}$$

$$\frac{\pi}{4} = \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3}$$

By using , we get  $\frac{\pi}{4} = \left\{ \frac{1}{2} - \frac{1}{3} \left(\frac{1}{2}\right)^3 + \frac{1}{5} \left(\frac{1}{2}\right)^5 - \dots \right\} + \left\{ \frac{1}{3} - \frac{1}{3} \left(\frac{1}{3}\right)^3 + \frac{1}{5} \left(\frac{1}{3}\right)^5 - \dots \right\}$

(3) *Hutton's series*

$$\frac{\pi}{4} = 3 \left\{ \frac{1}{4} - \frac{1}{3} \left(\frac{1}{4}\right)^3 + \frac{1}{5} \left(\frac{1}{4}\right)^5 - \dots \right\} + \left\{ \frac{5}{99} - \frac{1}{3} \left(\frac{5}{99}\right)^3 + \frac{1}{5} \left(\frac{5}{99}\right)^5 - \dots \right\}$$

*This series was discovered by Charles Hutton (1737 – 1823).*

*Proof:*  $\tan^{-1} a - \tan^{-1} b = \tan^{-1} \frac{a - b}{1 + ab}$  .....

Putting  $(a, b) = (1, \frac{1}{4}), (\frac{3}{5}, \frac{1}{4}), (\frac{7}{23}, \frac{1}{4})$  in , we get the following three equations.

$$\begin{aligned} \frac{\pi}{4} &= \tan^{-1} \frac{1}{4} + \tan^{-1} \frac{3}{5} \\ \tan^{-1} \frac{3}{5} &= \tan^{-1} \frac{1}{4} + \tan^{-1} \frac{7}{23} \\ \tan^{-1} \frac{7}{23} &= \tan^{-1} \frac{1}{4} + \tan^{-1} \frac{5}{99} \\ +) & \\ \hline \frac{\pi}{4} &= 3 \tan^{-1} \frac{1}{4} + \tan^{-1} \frac{5}{99} \end{aligned}$$

$$\frac{\pi}{4} = 3 \left\{ \frac{1}{4} - \frac{1}{3} \left(\frac{1}{4}\right)^3 + \frac{1}{5} \left(\frac{1}{4}\right)^5 - \dots \right\} + \left\{ \frac{5}{99} - \frac{1}{3} \left(\frac{5}{99}\right)^3 + \frac{1}{5} \left(\frac{5}{99}\right)^5 - \dots \right\}$$

(4) *Machin's series*

$$\frac{\pi}{4} = 4 \left\{ \frac{1}{5} - \frac{1}{3} \left(\frac{1}{5}\right)^3 + \frac{1}{5} \left(\frac{1}{5}\right)^5 - \dots \right\} - \left\{ \frac{1}{239} - \frac{1}{3} \left(\frac{1}{239}\right)^3 + \frac{1}{5} \left(\frac{1}{239}\right)^5 - \dots \right\}$$

*This series was discovered by John Machin (1680 – 1751) in 1706.*

*Proof:* Putting  $a = b = \frac{1}{5}$  in , we have

$$4 \tan^{-1} \frac{1}{5} = 2 \tan^{-1} \frac{5}{12} = \tan^{-1} \frac{\frac{5}{12} \times 2}{1 - \left(\frac{5}{12}\right)^2} = \tan^{-1} \frac{120}{119}$$

$$4 \tan^{-1} \frac{1}{5} - \tan^{-1} \frac{1}{239} = \tan^{-1} \frac{120}{119} - \tan^{-1} \frac{1}{239}$$

$$\begin{aligned}
 &= \tan^{-1} \frac{\frac{120}{119} - \frac{1}{239}}{1 + \frac{120}{119} \times \frac{1}{239}} = \tan^{-1} 1 = \frac{\mathbf{p}}{4} \\
 \frac{\mathbf{p}}{4} &= 4 \tan^{-1} \frac{1}{5} - \tan^{-1} \frac{1}{239} \\
 &= 4 \left\{ \frac{1}{5} - \frac{1}{3} \left(\frac{1}{5}\right)^3 + \frac{1}{5} \left(\frac{1}{5}\right)^5 - \dots \right\} - \left\{ \frac{1}{239} - \frac{1}{3} \left(\frac{1}{239}\right)^3 + \frac{1}{5} \left(\frac{1}{239}\right)^5 - \dots \right\}
 \end{aligned}$$

(5) *Stormer's series*  $\frac{\mathbf{p}}{4} = 6 \tan^{-1} \left(\frac{1}{8}\right) + 2 \tan^{-1} \left(\frac{1}{57}\right) + \tan^{-1} \left(\frac{1}{239}\right)$

*This series was discovered by Fredrik Carl Mulertz Stormer (1874 – 1957) in 1896.*

*Proof:*  $\tan^{-1} a + \tan^{-1} b + \tan^{-1} c = \tan^{-1} \frac{a + b + c - abc}{1 - (ab + bc + ca)} \dots\dots\dots$

Putting  $a = b = c = \frac{1}{8}$  in , we have

$$\begin{aligned}
 6 \tan^{-1} \frac{1}{8} &= 2 \tan^{-1} \frac{191}{488} \\
 6 \tan^{-1} \frac{1}{8} + 2 \tan^{-1} \frac{1}{57} &= 2 \tan^{-1} \frac{191}{488} + 2 \tan^{-1} \frac{1}{57} \\
 &= 2 \tan^{-1} \frac{91}{221} \\
 &= \tan^{-1} \frac{119}{120} \\
 6 \tan^{-1} \frac{119}{120} + 2 \tan^{-1} \frac{1}{57} + \tan^{-1} \frac{1}{239} &= \tan^{-1} \frac{119}{120} + \tan^{-1} \frac{1}{239} \\
 &= \tan^{-1} 1 = \frac{\mathbf{p}}{4}
 \end{aligned}$$

*Results of Pi calculation*

by using software, Mathcad 2000 Professional ( MathSoft,Inc. )

Gregory's

$$4 \sum_{k=1}^n \frac{(-1)^{k+1}}{2k-1}$$

Euler's

$$4 \sum_{k=1}^n \frac{(-1)^{k+1}(\frac{1}{2})^{2k-1}}{2k-1} + 4 \sum_{k=1}^n \frac{(-1)^{k+1}(\frac{1}{3})^{2k-1}}{2k-1}$$

*n=1* 4  
*n=500* 3.13959265558979  
*n=5000* 3.14141050422882  
*n=7000* 3.1414590709027

*n=1* 3.333333333333333  
*n=500* 3.14156158787759  
*n=1000* 3.14159267045069  
*n=1500* 3.14159265357837

Machin's

$$16 \sum_{k=1}^n \frac{(-1)^{k+1}(\frac{1}{5})^{2k-1}}{2k-1} - 4 \sum_{k=1}^n \frac{(-1)^{k+1}(\frac{1}{239})^{2k-1}}{2k-1}$$

*n=1* 3.18326359832636  
*n=90* 3.14159265362355

Stomer's

$$24 \sum_{k=1}^n \frac{(-1)^{k+1}(\frac{1}{8})^{2k-1}}{2k-1} + 8 \sum_{k=1}^n \frac{(-1)^{k+1}(\frac{1}{57})^{2k-1}}{2k-1} + 4 \sum_{k=1}^n \frac{(-1)^{k+1}(\frac{1}{239})^{2k-1}}{2k-1}$$

*n=1* 3.15708727886662      *n=7* 3.14159265358984

*References*

1. 円周率の歴史 平山 諦 著 1990 大阪教育図書  
 2 p の歴史 ペートル・ベックマン著田尾 陽一・清水 韶光訳 1973 蒼樹書房  
*Petr Beckmann, A History of PI, The Golem Press, 1971*