

The Verification of Archimedes' Pi

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1. Introduction

The famous genius Archimedes played a very active role in science in about 200 B.C. He calculated pi, which was correct to two places of decimals, 3.14.

It was so carefully calculated that a more accurate one could not be devised for 1,400 years after his death.

I verified his calculations in the interval of my study and admired his talent of calculation, accuracy, cautiousness and patience.

Without calculators, he had completed such complicated works. Thanks to a computer, I have rather easily done this work. I affixed the reference marks to the upper right of quotations and illustrations to trace their original ⁿ⁾ sources.

2. Inequality:

$$a \pm \frac{b}{2a \pm 1} < \sqrt{a^2 \pm b} < a \pm \frac{b}{2a} \dots\dots\dots(*)$$

According to some books, ^{1), 2), 3)} this inequality is used in ancient Greek times to get the value of \sqrt{A} after A is modified as follows:

$$A = a^2 \pm b \quad (b > 0)$$

Proof: Setting $\sqrt{a^2 \pm b} = a \pm c$ (a : positive integer c : positive decimal)

$$\begin{aligned} \text{we get} \quad a^2 \pm b &= a^2 \pm 2ac + c^2 > a^2 \pm 2ac \\ \pm b &> \pm 2ac \end{aligned}$$

$$\text{As } a > 0, \text{ then } \pm \frac{b}{2a} > \pm c$$

$$\sqrt{a^2 \pm b} = a \pm c < a \pm \frac{b}{2a} \dots\dots\dots$$

that is, $a^2 \pm b = (a \pm c)^2 = a^2 \pm 2ac + c^2$

As $0 < c < 1$, then $c^2 < c$.

$$a^2 \pm b < a^2 \pm 2ac + c = a^2 \pm c(2a \pm 1)$$

$$\pm b < \pm c(2a \pm 1)$$

As $a > 1$, then $2a \pm 1 > 0$

$$\pm \frac{b}{2a \pm 1} < \pm c$$

Therefore, $a \pm \frac{b}{2a \pm 1} < a \pm c = \sqrt{a^2 \pm b} \dots\dots\dots$

$$a \pm \frac{b}{2a \pm 1} < \sqrt{a^2 \pm b} < a \pm \frac{b}{2a} \quad \{ \text{by using } \dots \text{ and } \dots \}$$

Note: They say that the ancient Babylonians approximated \sqrt{A} as follows:

$$\sqrt{a^2 + b} \approx a + \frac{b}{2a} \quad ^{1)}$$

Proof:
$$\sqrt{a^2 + b} = \sqrt{a \times \frac{a^2 + b}{a}} \quad (\text{geometric mean})$$

$$\frac{a + \frac{a^2 + b}{a}}{2} \quad (\text{arithmetic mean})$$

$$= a + \frac{b}{2a}$$

But the equality holds if and only if $b = 0$.

3. Approximating square roots by Archimedes

There is no descriptions how Archimedes approximated square roots except for its results. However, some historians speculate that he might use the above mentioned inequality (*). Let us try the calculation with this inequality.

(1) Approximating $\sqrt{3}$ ^{1),3)}

$$(a) \frac{5}{3} < \sqrt{3} < \frac{7}{4}$$

We set 2 for approximate value of $\sqrt{3}$ because $\sqrt{3}$ is between 1 and $\sqrt{4}$. By using inequality (*), let $\sqrt{3} = \sqrt{2^2 - 1}$ so that

$$2 - \frac{1}{2 \times 2 - 1} < \sqrt{2^2 - 1} < 2 - \frac{1}{2 \times 2}$$

$$\frac{5}{3} < \sqrt{3} < \frac{7}{4}$$

$$(b) \frac{19}{11} < \sqrt{3} < \frac{26}{15}$$

By using (a), we get $\frac{5 \times 3}{3} < 3\sqrt{3} < \frac{7 \times 3}{4}$; that is, $5 < \sqrt{27} < \frac{21}{4}$.

So, we are able to set 5 for approximate value of $\sqrt{27}$.

By using (*), let $\sqrt{27} = \sqrt{5^2 + 2}$ so that we get

$$5 + \frac{2}{2 \times 5 + 1} < \sqrt{5^2 + 2} < 5 + \frac{2}{2 \times 5}$$

$$\text{leading to } \frac{57}{11} < 3\sqrt{3} < \frac{26}{5}$$

$$\frac{19}{11} < \sqrt{3} < \frac{26}{15}$$

$$(c) \frac{265}{153} < \sqrt{3} < \frac{1351}{780} \dots\dots\dots (a)$$

By using (b), we get $\frac{19 \times 15}{11} < 15\sqrt{3} < \frac{26 \times 15}{15}$; that is, $\frac{285}{11} < \sqrt{675} < 26$.

So, we are able to set 26 for approximate value of $\sqrt{675}$.

By using (c), let $\sqrt{675} = \sqrt{26^2 - 1}$ so that we get

$$26 - \frac{1}{2 \times 26 - 1} < \sqrt{26^2 - 1} < 26 - \frac{1}{2 \times 26}$$

$$\text{leading to } \frac{1325}{51} < 15\sqrt{3} < \frac{1351}{52}$$

$$\frac{265}{153} < \sqrt{3} < \frac{1351}{780}$$

Note: 1) Approximation is made to both sides alternately.

2) $\sqrt{3}$ is correct to 4 places of decimals; $1.7320261 < \sqrt{3} < 1.732052$.

(2) $\sqrt{349452} > 591\frac{1}{8}$ (b)

$$\begin{aligned} \sqrt{349452} &= \sqrt{591^2 + 169} \\ &> 591 + \frac{169}{2 \times 591 + 1} \\ &= 591 + \frac{169}{1183} \\ &= 591\frac{1}{7} \\ &> 591\frac{1}{8} \end{aligned}$$

Note: He selected $\frac{1}{8}$ rather than $\frac{1}{7}$.

(3) $\sqrt{1373943\frac{33}{64}} > 1172\frac{1}{8}$ (c)

$$\begin{aligned} \sqrt{1373943\frac{33}{64}} &= \frac{1}{8}\sqrt{87932385} \\ &= \frac{1}{8}\sqrt{9377^2 + 4256} \\ &> \frac{1}{8}\left(9377 + \frac{4256}{2 \times 9377 + 1}\right) \\ &= \frac{1}{8}\left(9376 + \frac{23011}{18755}\right) \\ &> 1172\frac{1}{8} \end{aligned}$$

$$\begin{aligned}
 (4) \quad \sqrt{5472132 \frac{1}{16}} &> 2339 \frac{1}{4} && \dots\dots\dots (d) \\
 \sqrt{5472132 \frac{1}{16}} &= \frac{1}{4} \sqrt{87554113} \\
 &= \frac{1}{4} \sqrt{9357^2 + 664} \\
 &> \frac{1}{4} \left(9357 + \frac{664}{2 \times 9357 + 1} \right) \\
 &= \frac{1}{4} \left(9356 + \frac{19379}{18715} \right) \\
 &> 2339 \frac{1}{4}
 \end{aligned}$$

$$\begin{aligned}
 (5) \quad \sqrt{9082321} &< 3013 \frac{3}{4} && \dots\dots\dots (e) \\
 \sqrt{9082321} &= \sqrt{3014^2 - 1875} \\
 &< 3014 - \frac{1875}{2 \times 3014} \\
 &= 3014 - \frac{1875}{6028} \\
 &= 3014 - \frac{1}{4} \\
 &= 3013 \frac{3}{4}
 \end{aligned}$$

Note: He selected $\frac{1}{4}$.

$$\begin{aligned}
 (6) \quad \sqrt{3380929} &< 1838 \frac{9}{11} && \dots\dots\dots (f) \\
 \sqrt{3380929} &= \sqrt{1839^2 - 992} \\
 &< 1839 - \frac{992}{2 \times 1839} \\
 &= 1838 \frac{2686}{3678} \\
 &< 1838 \frac{9}{11}
 \end{aligned}$$

Note: He selected $\frac{9}{11}$ rather than $\frac{3}{4}$.

(7) $\sqrt{1018405} < 1009\frac{1}{6}$ (g)

$$\begin{aligned} \sqrt{1018405} &= \sqrt{1009^2 + 324} \\ &< 1009 + \frac{324}{2 \times 1009} \\ &< 1009\frac{1}{6} \end{aligned}$$

(8) $\sqrt{4069284\frac{1}{36}} < 2017\frac{1}{4}$ (h)

$$\begin{aligned} \sqrt{4069284\frac{1}{36}} &= \frac{1}{6} \sqrt{146494225} \\ &= \frac{1}{6} \sqrt{12102^2 + 35821} \\ &< \frac{1}{6} \left(12102 + \frac{35821}{2 \times 12102} \right) \\ &= \frac{1}{6} \left(12102 + \frac{35821}{24204} \right) \\ &< \frac{1}{6} \left(12102 + \frac{3}{2} \right) = 2017\frac{1}{4} \end{aligned}$$

4. Measuring pi

There are fifteen writings which Archimedes wrote. ^{1),2),3),4)} "The Measurement of the Circle" is one of them. Referring to circumference of circle, it says as follows. "It is shorter than three times of its diameter and an excess, one-seventh of its diameter" and "It is longer than three times of its diameter and an excess, ten seventy-firsts of its diameter."

This means $3\frac{10}{71} < p < 3\frac{1}{7}$, that is, $3.1408 < p < 3.1429$.

We realize that he calculated pi correctly to two places of decimals.

(1) An outline of the calculation

To make a start, we draw a circumscribed regular hexagon to a given circle O. Doubling its sides of circumscribed regular polygons, from hexagon, dodecahedron, 24, 48 to a 96 sided-polygon, we find that each perimeter is decreasingly getting closer and closer to $3\frac{1}{7}$ of the diameter.

Similarly, we draw a inscribed regular hexagon to a circle O. Doubling its sides of inscribed regular polygon to a 96 sided regular polygon, we find that each perimeter is, increasingly, getting closer and closer to $3\frac{10}{71}$ of the diameter.

$$3\frac{10}{71} \times (\text{Diameter of circle O}) < p \times (\text{Diameter of circle O}) < 3\frac{1}{7} \times (\text{Diameter of circle O})$$

We conclude that $3\frac{10}{71} < p < 3\frac{1}{7}$.

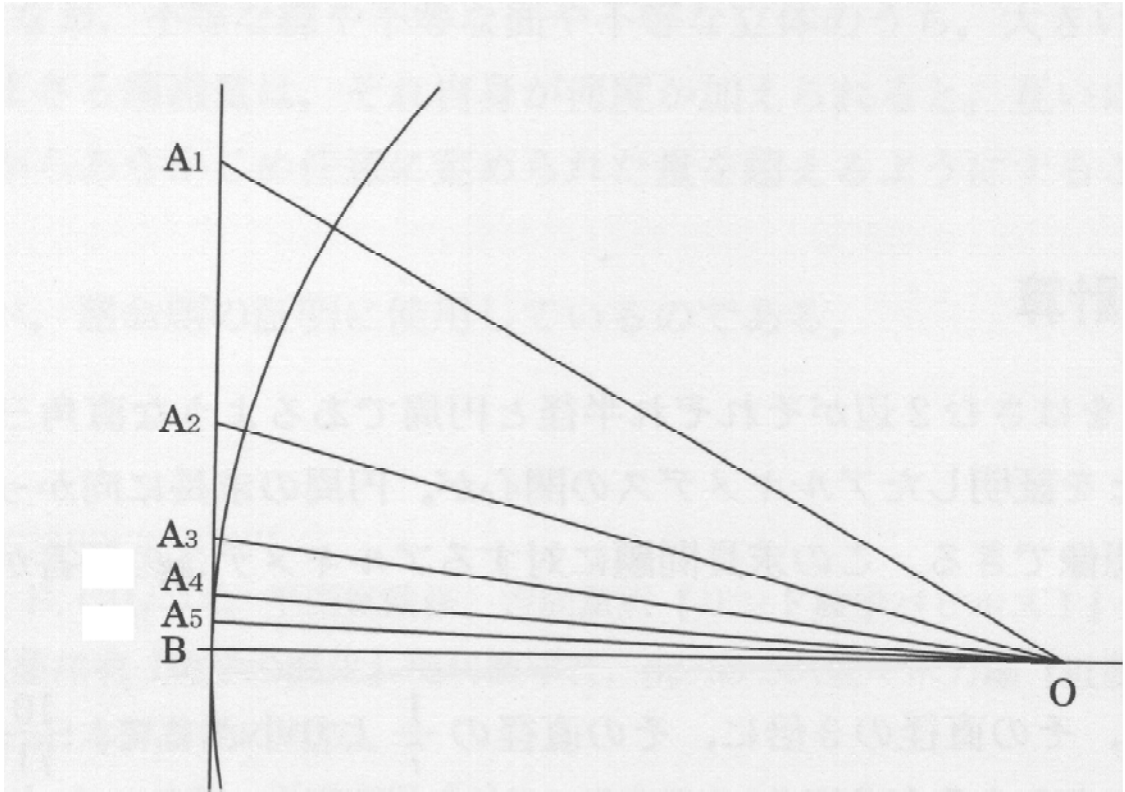
Let us verify his calculations.

(2) Calculating perimeter

(i) Circumscribed 96 sided-polygon

The next figure shows a circle with a center O and a radius OB; and the parts

of five regular polygons circumscribed out of circle at a point B.



- A_1B ... a half side of regular circumscribed hexagon out of the circle
- A_2B ... a half side of regular circumscribed dodecahedron out of the circle
- A_3B ... a half side of regular circumscribed 24 sided polygon out of the circle
- A_4B ... a half side of regular circumscribed 48 sided polygon out of the circle
- A_5B ... a half side of regular circumscribed 96 sided polygon out of the circle

Because A_1OB is a half part of regular triangle, by using (a), we get

$$\frac{BO}{A_1B} = \sqrt{3} > \frac{265}{153}, \quad \frac{A_1O}{A_1B} = 2 \dots\dots\dots$$

Because OA_1 is a bisector of A_1OB , we get

$$\begin{aligned}
 (\cdot) \quad \frac{A_1O}{BO} &= \frac{A_1A_2}{A_2B} & \frac{A_1O + BO}{BO} &= \frac{A_1A_2 + A_2B}{A_2B} & (\text{componendo}) \\
 & & \frac{BO}{A_2B} &= \frac{A_1O + BO}{A_1B} \\
 & & &= \frac{A_1O}{A_1B} + \frac{BO}{A_1B}
 \end{aligned}$$

$$\begin{aligned}
 (\because) \quad \left(\frac{A_2O}{A_2B}\right)^2 &= \frac{A_2B^2 + BO^2}{A_2B^2} & (\text{Pythagorean theorem}) & \quad \text{Note: Let } A_2 = h, \text{ then} \\
 &= 1 + \left(\frac{BO}{A_2B}\right)^2 & & \quad \left(\frac{1}{\cos h}\right)^2 = 1 + \tan^2 h
 \end{aligned}$$

The following results hold in general.

$$\begin{aligned}
 (\cdot) \quad \frac{BO}{A_nB} &= \frac{A_{n-1}O}{A_{n-1}B} + \frac{BO}{A_{n-1}B} \\
 (\because) \quad \left(\frac{A_nO}{A_nB}\right)^2 &= 1 + \left(\frac{BO}{A_nB}\right)^2
 \end{aligned}$$

$$\begin{aligned}
 (\cdot) \quad \frac{BO}{A_2B} &= \frac{A_1O}{A_1B} + \frac{BO}{A_1B} & \{ \text{by using } \} \\
 &> \frac{265}{153} + 2 \\
 &= \frac{571}{153}
 \end{aligned}$$

$$\begin{aligned}
 (\because) \quad \left(\frac{A_2O}{A_2B}\right)^2 &= 1 + \left(\frac{BO}{A_2B}\right)^2 \\
 &> 1 + \left(\frac{571}{153}\right)^2 \\
 &= \frac{349450}{153^2}
 \end{aligned}$$

$$\frac{A_2O}{A_2B} > \frac{\sqrt{349450}}{153} \quad \{ \text{by using (b)} \}$$

$$> \frac{591\frac{1}{8}}{153}$$

$$\frac{BO}{A_2B} > \frac{571}{153}, \quad \frac{A_2O}{A_2B} > \frac{591\frac{1}{8}}{153} \dots\dots\dots$$

$$(\cdot) \quad \frac{BO}{A_3B} = \frac{A_2O}{A_2B} + \frac{BO}{A_2B} \quad \{by \text{ using } \}$$

$$> \frac{591\frac{1}{8}}{153} + \frac{571}{153}$$

$$= \frac{1162\frac{1}{8}}{153}$$

$$(:) \quad \left(\frac{A_3O}{A_3B}\right)^2 = 1 + \left(\frac{BO}{A_3B}\right)^2$$

$$> 1 + \left(\frac{1162\frac{1}{8}}{153}\right)^2$$

$$= \frac{137394\frac{33}{64}}{153^2}$$

$$\frac{A_3O}{A_3B} > \frac{\sqrt{137394\frac{33}{64}}}{153} \quad \{by \text{ using } (c)\}$$

$$> \frac{1172\frac{1}{8}}{153}$$

$$\frac{BO}{A_3B} > \frac{1162\frac{1}{8}}{153}, \quad \frac{A_3O}{A_3B} > \frac{1172\frac{1}{8}}{153} \dots\dots\dots$$

$$\begin{aligned}
 (\cdot) \quad \frac{BO}{A_4B} &= \frac{A_3O}{A_3B} + \frac{BO}{A_3B} && \{by\ using\ \} \\
 &> \frac{1162\frac{1}{8}}{153} + \frac{1172\frac{1}{8}}{153} \\
 &= \frac{2334\frac{1}{8}}{153}
 \end{aligned}$$

$$\begin{aligned}
 (\cdot) \quad \left(\frac{A_4O}{A_4B}\right)^2 &= 1 + \left(\frac{BO}{A_4B}\right)^2 \\
 &> 1 + \left(\frac{2334\frac{1}{4}}{153}\right)^2 \\
 &= \frac{5471632\frac{1}{16}}{153^2}
 \end{aligned}$$

$$\begin{aligned}
 \frac{A_4O}{A_4B} &> \frac{\sqrt{5471632\frac{1}{16}}}{153} && \{by\ using\ (d)\} \\
 &> \frac{2339\frac{1}{4}}{153}
 \end{aligned}$$

$$\frac{BO}{A_4B} > \frac{2334\frac{1}{4}}{153}, \quad \frac{A_4O}{A_4B} > \frac{2339\frac{1}{4}}{153} \dots\dots\dots$$

$$\begin{aligned}
 (\cdot) \quad \frac{BO}{A_5B} &= \frac{A_4O}{A_4B} + \frac{BO}{A_4B} && \{by\ using\ \} \\
 &> \frac{2339\frac{1}{4}}{153} + \frac{2334\frac{1}{4}}{153} \\
 &= \frac{4673\frac{1}{2}}{153}
 \end{aligned}$$

$$A_5B < \frac{153}{4673\frac{1}{2}} BO \dots\dots\dots$$

Setting L as the perimeter of 96-sided circumscribed polygon, then we get the following results.

$$\begin{aligned} L &= 2 \times A_5B \times 96 \quad \{ \text{by using } \} \\ &< 2 \times \frac{153}{4673\frac{1}{2}} OB \times 96 \\ &= 3 \frac{1335}{9347} \times (2BO) \\ &< 3\frac{1}{7} \times (2BO) \end{aligned}$$

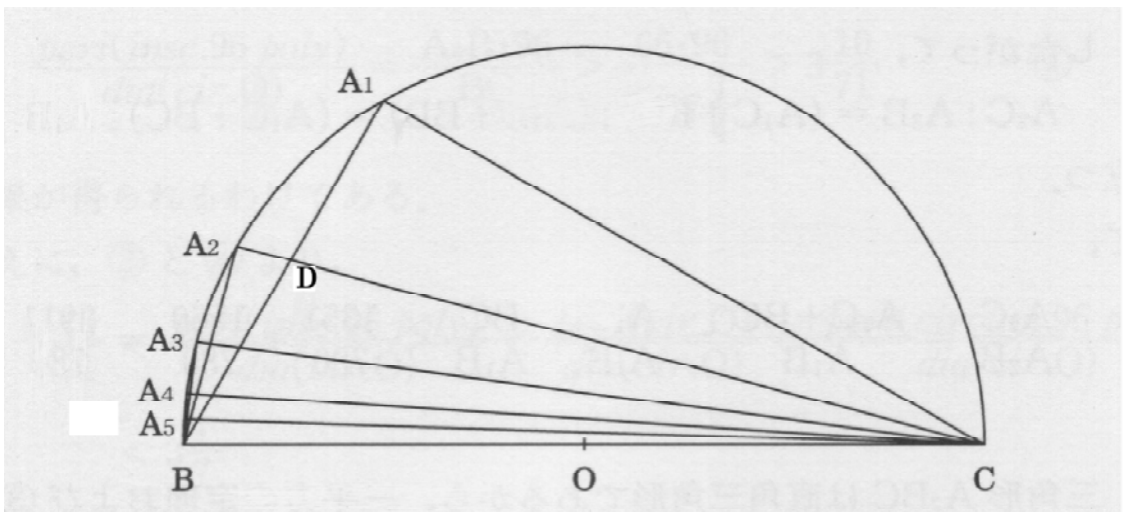
As the perimeter of circle is $p \times (2BO)$,

$$p \times (2OB) < 3\frac{1}{7} \times (2OB)$$

$$p < 3\frac{1}{7} \dots\dots\dots(*) (*)$$

(ii) Inscribed 96-sided regular polygon

The next figure shows a circle with a center O and a radius OB , and parts of five regular polygons inscribed in the circle at a point B .



$A_1B...$ a side of regular inscribed hexagon in the circle

$A_2B...$ a side of regular inscribed dodecahedron in the circle

$A_3B...$ a side of regular inscribed 24-sided polygon in the circle

$A_4B...$ a side of regular inscribed 48-sided polygon in the circle

$A_5B...$ a side of regular inscribed 96-sided polygon in the circle

$$A_2BC \quad A_1DC \quad A_2BD$$

$$\frac{A_2C}{A_2B} = \frac{A_1C}{A_1D} = \frac{A_2B}{A_2D} \dots\dots\dots$$

Because DC is a bisector of C in A_1BC ,

$$\frac{BC}{A_1C} = \frac{BD}{A_1D} \quad \frac{A_1C}{A_1D} = \frac{BC}{BD} \dots\dots\dots$$

By using and

$$\frac{A_2C}{A_2B} = \frac{A_1C}{A_1D} = \frac{BC}{BD} \dots\dots\dots$$

$$(\cdot) \quad \frac{A_2C}{A_2B} = \frac{A_1C + BC}{A_1D + BD} \quad (\text{componendo})$$

$$= \frac{A_1C + BC}{A_1B}$$

$$(\cdot) \quad \left(\frac{BC}{A_2B} \right)^2 = \frac{A_2B^2 + A_2C^2}{A_2B^2} = 1 + \left(\frac{A_2C}{A_2B} \right)^2$$

Note: Let $A_2 = h$,
 then $\left(\frac{1}{\cos h} \right) = 1 + \tan^2 h$

The following results hold in general.

$$(\cdot) \quad \frac{A_nC}{A_nB} = \frac{A_{n-1}C}{A_{n-1}B} + \frac{BC}{A_{n-1}B}$$

$$(\cdot) \quad \left(\frac{BC}{A_nB} \right)^2 = 1 + \left(\frac{A_nC}{A_nB} \right)^2$$

Because A_1BO is a regular triangle,

$$\frac{A_1C}{A_1B} = \sqrt{3} < \frac{1351}{7801}, \quad \frac{BC}{A_1B} = 2 \dots\dots\dots$$

$$\begin{aligned}
 (\bullet) \quad \frac{A_2C}{A_2B} &= \frac{A_1C}{A_1B} + \frac{BC}{A_1B} \\
 &< \frac{1351}{780} + 2 \\
 &= \frac{2911}{780}
 \end{aligned}$$

$$\begin{aligned}
 (\because) \quad \left(\frac{BC}{A_2B}\right)^2 &= 1 + \left(\frac{A_2C}{A_2B}\right)^2 \\
 &< 1 + \left(\frac{2911}{780}\right)^2
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{9082321}{608400} \\
 \frac{BC}{A_2B} &= \frac{\sqrt{9082321}}{780} \quad \{by\ using\ (e)\} \\
 &< \frac{3013\frac{3}{4}}{780}
 \end{aligned}$$

$$\frac{A_2C}{A_2B} < \frac{2911}{780}, \quad \frac{BC}{A_2B} < \frac{3013\frac{3}{4}}{780} \quad \dots\dots\dots$$

$$\begin{aligned}
 (\bullet) \quad \frac{A_3C}{A_3B} &= \frac{A_2C}{A_2B} + \frac{BC}{A_2B} \\
 &< \frac{2911}{780} + \frac{3013\frac{3}{4}}{780} \\
 &= \frac{1823}{240}
 \end{aligned}$$

$$\begin{aligned}
 (\because) \quad \left(\frac{BC}{A_3B}\right)^2 &= 1 + \left(\frac{A_3C}{A_3B}\right)^2 \\
 &< 1 + \left(\frac{1823}{240}\right)^2
 \end{aligned}$$

$$= \frac{3380929}{240^2}$$

$$\frac{BC}{A_3B} < \frac{\sqrt{3380929}}{240} \quad \{by\ using\ (f)\}$$

$$< \frac{1838\frac{9}{11}}{240}$$

$$\frac{A_3C}{A_3B} < \frac{1823}{240}, \quad \frac{BC}{A_3B} < \frac{1838\frac{9}{11}}{240} \quad \dots\dots\dots$$

$$(\cdot) \quad \frac{A_4C}{A_4B} = \frac{A_3C}{A_3B} + \frac{BC}{A_3B}$$

$$< \frac{1823}{240} + \frac{1838\frac{9}{11}}{240}$$

$$= \frac{3661\frac{9}{11}}{240}$$

$$= \frac{1007}{66}$$

$$(:) \quad \left(\frac{BC}{A_4B}\right)^2 = 1 + \left(\frac{A_4C}{A_4B}\right)^2$$

$$< 1 + \left(\frac{1007}{66}\right)^2$$

$$= \frac{1018405}{66^2}$$

$$\frac{BC}{A_4B} < \frac{\sqrt{1018405}}{66} \quad \{by\ using\ (g)\}$$

$$< \frac{1009\frac{1}{6}}{66}$$

$$\frac{A_4C}{A_4B} < \frac{1007}{66}, \quad \frac{BC}{A_4B} < \frac{1009\frac{1}{6}}{66} \dots\dots\dots$$

$$\begin{aligned} (\cdot) \quad \frac{A_5C}{A_5B} &= \frac{A_4C}{A_4B} + \frac{BC}{A_4B} \\ &< \frac{1007}{66} + \frac{1009\frac{1}{6}}{66} \\ &= \frac{2016\frac{1}{6}}{66} \end{aligned}$$

$$\begin{aligned} (\because) \quad \left(\frac{BC}{A_5B}\right)^2 &= 1 + \left(\frac{A_5C}{A_5B}\right)^2 \\ &< 1 + \left(\frac{2016\frac{1}{6}}{66}\right)^2 \\ &= \frac{4069284\frac{1}{36}}{66^2} \end{aligned}$$

$$\begin{aligned} \frac{BC}{A_5B} &< \frac{\sqrt{4069284\frac{1}{36}}}{66} && \{by \text{ using (h)}\} \\ &< \frac{2017\frac{1}{4}}{66} \\ \frac{66 \times BC}{2017\frac{1}{4}} &= A_5B \end{aligned}$$

Because the perimeter of 96-sided regular inscribed polygon is $p \times BC$, we have

$$\begin{aligned}
p \times BC &= A_5B \times 96 \\
&> \frac{6336}{2017\frac{1}{4}} BC \\
&= \frac{6336}{2017\frac{1}{4}} BC \\
&= 3\frac{1137}{8069} BC \\
&> 3\frac{10}{71} BC \\
p &> 3\frac{10}{71} \quad \dots\dots\dots(***)
\end{aligned}$$

By using (**) and (***),

$$3\frac{10}{71} < p < 3\frac{1}{7}$$

Reference Books

- | | | | |
|---|---------|------|--------|
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